IDENTITIES FOR HYPERBOLIC SURFACES.

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OUTLINE

0: Introduction

1: Identities, generalizations & applications

2: Sketch of proofs of Identities of Basmajian, McShane, & Bridgeman

3: Sketch of proof for identity for closed surfaces (Luo-T).

4: Further directions.
0. Introduction:

Topology: Surfaces, with/without boundary

Geometry: (marked) hyperbolic structures on $S$

(Representations: $\rho: \pi_1(S) \to G$, $\mathcal{Y}_{g,n}$, $\mathcal{M}_{g,n}$)

Teichmüller space $\approx$ moduli space

Analysis: Various identities on hyperbolic surfaces, respecting group actions.
Introduction...

- Give a new identity for (closed) hyperbolic surfaces.
- Relate various identities for surfaces with boundary by Basmajian, McShane, Bridgemen & various others.
- Discuss some generalizations & applications (from last few years...)
- Unified point of view & sketch proofs for the various identities
- Sketch proof of identity for closed surfaces.
1/ Identities

A. Thm (Basmajian ’93 AJM)

\[ \sum_{\alpha} B(1 \cdot 1) = L \ \text{Total length of } DS \]

over all oriented orthogeodesics

\[ B(x) = 2 \log \coth \frac{x}{2} = 2 \sinh^{-1}(\frac{1}{\sinh x}) \]
B1 Thm (McShane '91, 98)

\[ 2 \sum_{\alpha, \beta} \frac{1}{1 + \exp \left( \frac{|\alpha| + |\beta|}{2} \right)} = 1. \]

\[ M(x, y) = \frac{2}{1 + \exp \left( \frac{x + y}{2} \right)} \]

over all \( \alpha, \beta \)

cusp
horocycle of length 1
B2 Thm (Mirzakhani '08, T-Wong-Zhang '07)

\[ 2 \sum_{x, \beta} G \left( \frac{1}{2}, \frac{|w|}{2}, \frac{|p|}{2} \right) = \text{length of } \partial \Delta = \partial S \]

\[ G(x, y, z) = \log \left( \frac{e^x + e^{y+z}}{e^{-x} + e^{y+z}} \right) \]

Here \( \partial S = \delta \) with \( |\delta| = L \) and \( \delta \) may be geodesic boundary or cone pt. of \( \alpha \) for \( 0 \leq \theta \leq \pi \).
Theorem (Bridgeman '11)

\[ \sum_{\gamma} \text{Br}(\gamma) = -4\pi^2 \chi(S) \]

\[ \text{Vol}(T, (S)) \]

over all oriented orthogeodesics

\[ \text{Br}(x) = 4 \mathcal{L} \left( \frac{1}{\cosh^2 \frac{x}{2}} \right) \]

where \( \mathcal{L}(x) \) is Roger's dilog function

\[
\mathcal{L}(0) = 0, \quad -2 \mathcal{L}'(x) = \frac{\log x}{1-x} + \frac{\log(1-x)}{x}, \quad 0 \leq x \leq 1
\]
\[ \sum_P f(P) + \sum_T g(T) = -4\pi^2 \chi(S) \]

over all p.o.p. \hspace{1cm} over all one-holed tori
\[ f(P) = 4 \sum_{i \neq j} \left[ 2 \mathcal{L}\left( \frac{1-x_i}{1-x_i y_j} \right) - 2 \mathcal{L}\left( \frac{1-y_i}{1-x_i y_j} \right) - \mathcal{L}(y_j) - \mathcal{L}\left( \frac{(1-x_i)^2 y_j}{(1-y_j)^2 x_i} \right) \right] \]

\[ x_i = e^{-l_i}, \quad y_j = \tanh^2 \left( \frac{m_j}{2} \right) \]

\[ g(T) = 4\pi^2 + 8 \sum_A \left[ 2 \mathcal{L}\left( \frac{1-x_A}{1-x_A y_A} \right) - 2 \mathcal{L}\left( \frac{1-y_A}{1-x_A y_A} \right) - 2 \mathcal{L}(y_A) - \mathcal{L}\left( \frac{(1-x_A)^2 y_A}{(1-y_A)^2 x_A} \right) \right] \]

\[ x_A = e^{-\alpha}, \quad y_A = \tanh^2 \left( \frac{m_A}{2} \right) \]

\[ |A| = A \]
Generalizations & Applications

A - Basmajian - higher dimensions

B - McShane - geodesic boundaries, cone singularities (Mirzakhani, T-Wong-Zhang), punctured surface bundles over circle (Bowditch, Akiyoshi-Miyachi-Sakuma), quasi-fuchsian reps (Bowditch), hyperbolic 3-mfds from Dehn Surgery (T-Wong-Zhang), closed genus 2 surface (McShane), Hitchin components (Labourie-McShane), complex hyperbolic reps (Kim-Kim-T), Do-Norbury, 2-bridge knots (Lee-Sakuma)....

Applications - (Mirzakhani) WP-vol of moduli space, asymptotics of growth of lengths of simple closed geodesics, Kontsevich-Witten Thm etc....
C - Bridgeman - higher dimensions, (Bridgeman- Kahn), applications to lower bounds for volumes.

D - Luo-T: Any number (including none) of boundary components. Possible applications to the counting of embedded pairs of pants.

E - Basmajian & McShane as the zeroth & first moment of the set of geodesics (with the Liouville measure) endowed with the length function. (Bridgeman- T).

E - Families of identities for cusped surfaces generalising the Basmajian & Bridgeman identities which hold for surfaces with cusps. (Basmajian- Parlier- T).

G - Generalizations of the identities for non-orientable surfaces (Luo-T, Huang- Norbury,...) real projective surfaces & Hitchin representations (Labovic-McShane, Vlamis-Yamada), complex-hyperbolic representations (Kim-Kim-T), .........
2. **Sketch of proofs.**

**Idea:** \((X, \mu), \text{ with } \mu(X) < \infty.\)

A. Find interesting geometric & measure theoretic decompositions of \(X\).

\[
X = \mathbb{Z} \cup \bigcup_{i} W_i \\
\mu(\mathbb{Z}) = 0
\]

\[
\Rightarrow \mu(X) = \sum_{i} \mu(W_i).
\]

B. Compute \(\mu(W_i)\) to obtain an identity.
Example - 1:

\[ X = [0, 1], \quad Z = \{ 0, \frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \ldots \} \cup \{ 1 \} \]

\[ \uparrow \text{countable} \]

\( W_i \) are complementary open intervals, ordered from left to right.

\[ W_1 = (0, \frac{1}{2}), \quad W_2 = (\frac{1}{2}, \frac{3}{4}), \ldots \]

\[ \mu(X) = \mu(Z) + \sum_{i=1}^{\infty} \mu(W_i) \]

\[ \Rightarrow 1 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots = \sum_{i=1}^{\infty} \frac{1}{2^i} \]
Example 0:

\[ X = [0,1], \ Z = \text{Cantor Set obtained from removing middle } \frac{1}{3} \text{ construction.} \]

\[ W_i = \text{complementary intervals - ordered by length } \alpha \text{ from left to right.} \]

\[ \mu(X) = \mu(Z) + \sum_{i=1}^{\infty} \mu(W_i) \]

\[ \Rightarrow 1 = 0 + \frac{1}{3} + \frac{2}{3^2} + \ldots + \frac{2^i}{3^{i+1}} + \ldots = \sum_{i=1}^{\infty} \frac{2^{i-1}}{3^i} . \]

In fact, Basmajian & McShane are interesting variations of example 0!
Example 1. (Rasmajian).

\[ X = \{ \text{unit tangent vectors on } \partial S \perp \nu \text{ to } \partial S \} \cong \partial S. \]

\[ \mu(x) = L = \text{length} (\partial S). \]

\[ \nu \in X, \quad \gamma(v) = \text{maximal geodesic in } S \]

obtained by exponentiating \( \nu \).

\[ \gamma(v): [0, T] \to S, \quad \gamma(v)(t) = \exp (t \nu). \text{ (laser ray starting from } \nu) \]

\{d\lambda\} \text{ set of oriented orthogeodesics on } S.

Q: When is \( T = \infty \) & how to decompose \( X \)?
Answer: \( Z = \{ v \in X \mid T = \infty \} \) has measure zero (limit set has measure 0)

Every \( v \in X \setminus Z \) is homotopic rel. to \( dS \) to a unique orthogeodesic \( \alpha_i \).

Define \( W(\alpha_i) = \{ v \in X \mid \gamma(v) \sim \alpha_i \} \), then \( X = Z \cup \bigsqcup W(\alpha_i) \)

\[ \Rightarrow L = \sum_i m(W(\alpha_i)) \]

\[ m(W(\alpha_i)) = 2 \log \coth \frac{\omega_i}{2} \] by elementary hyperbolic geometry calculations.
Example 2 (McShane)

(For simplicity), suppose $\partial S$ has 1 geodesic boundary component.
Again, $X = \{\text{unit tangent vectors on } \partial S \perp \text{ to } \partial S \} \cong \partial S$, $\mu(X) = L$

! Instead of losers, build walls by exponentiating $v$.

$G(v) : [0, T] \rightarrow S$ s.t. $G(v)(t) = \exp(tv)$ & $G(v)$ is an embedding on $[0, T)$. 
\[ Z = \{ v \in X \mid G(v) \text{ is infinite, i.e. } T = \infty \}. \]

Again \( \mu(Z) = 0 \) (when using surfaces with cusps only, need Birman-Series).

For \( v \in X \setminus Z \), "stability", nearby \( v \) generate 'homotopic' \( G(v) \).

Namely: if \( v \in X \setminus Z \), \( N(\partial S \cup G(v)) \) is a pair of pants \( P \) which can be made "geometric", so \( \partial P = \partial S \cup \alpha \cup \beta \) where \( \alpha, \beta \) disjoint simple closed geodesics.

Let \( \{ P \} \) be set of embedded p.o.p.'s in \( S \) whose boundary includes \( \partial S \), i.e. let

\[ W(P) = \{ v \in X \mid N(G(v) \cup \partial S) \sim P \}, \]

then \( X = Z \cup \bigsqcup P W(P) \).

Computing \( \mu(W(P)) \) gives \( G(x,y,z) \). Note: computation is 'localized' at \( P \).
Example 3 (Bridgeman)

\[ X = T_1(\Sigma) \] with the invariant measure, so \( \mu(X) = \text{vol}(T_1(\Sigma)) = -4\pi^2 \chi(\Sigma) \).

As with Basmajian, we use lasers and generate in both directions.

\( v \in X = T_1(\Sigma) \), define \( \gamma(v) \) to be the maximal geodesic

\[ \gamma(v) : [T_1, T_2] \rightarrow \Sigma \quad \text{s.t.} \quad \gamma(v)(t) = \exp(tv). \quad (T_1 < 0 < T_2) \]

Q: When is \( \gamma(v) \) infinite, i.e. \( T_1 = -\infty \) or \( T_2 = \infty \)?

What happens when \( \gamma(v) \) is finite (so \( \gamma(T_1), \gamma(T_2) \in \partial\Sigma \))?
Let \( Z = \{ v \in T_1(S) \mid \gamma(v) \text{ is infinite} \} \). Then by ergodicity of the geodesic flow, \( \mu(Z) = 0 \).

If \( v \in T_1(S) \setminus Z \), then \( \gamma(v) \) is homotopic rel. to \( \partial S \) to a unique oriented orthogeodesic in \( S \), \( \gamma(v) \sim \alpha_i \) for some \( i \).

Let \( W(\alpha_i) = \{ v \in T_1(S) \mid \gamma(v) \sim \alpha_i \} \).

Then \( T_1(S) = Z \sqcup \bigsqcup_{i} W(\alpha_i) \Rightarrow \mu(T_1(S)) = \sum_{i} \mu(W(\alpha_i)) \uparrow \)

\[ = B_e(1\alpha, 1). \]
Bridgeman's function:

Computing $\mu(W(x_i))$.

Calculate the measure of all $v \in T_1(\mathbb{H}^2)$ s.t. $G(v)$ begins at $A$ and ends at $B$. 

\( H^2 \)
Example 4 (Identity for closed surfaces, Luo-T).

What to do if $\partial S = \emptyset$?

Key ideas: Start everywhere in every direction & build walls.
\[ X = T_1(S), \quad m(X) = -4\pi^2 \chi(S). \]

For \( v \in T_1(S) \), \( G(v) \) = geodesic obtained by building maximal wall at equal speed in both directions.
$G(\nu) : [T_1, T_2] \to S$, $G(\nu)(t) = \exp(t \nu)$, $G(\nu)$ is embedding on $(T_1, T_2)$.

- $Z = \{ \nu \in T_1(S) \mid G(\nu) \text{ is infinite} \}$ has measure 0 by Birman-Series.

- $\nu \in T_1(S) \setminus Z \implies G(\nu)$ is an embedded graph of euler characteristic $-1$.

- $N(G(\nu))$ is a surface of euler characteristic $-1$, i.e. $P$ or $T$.

- $P$ or $T$ can be made 'geometric' (geodesic boundary) if $\nu \in T_1(P)$ or $T_1(T)$.

Define $W(P) = \{ \nu \in T_1(S) \mid G(\nu) \text{ is a spine for } P \}$ for $P$

or $W(T) = \{ \nu \in T_1(S) \mid G(\nu) \text{ is a spine for } T \}$ for $T$
Then
\[ \text{Vol}(T, \mathbf{s}) = -4\pi^2 \chi(\mathbf{s}) = \mathcal{O} + \sum_p \mu(W(p)) + \sum_T \mu(W(T)). \]

The functions \( f \) and \( g \) are \( f(p) = \mu(W(p)), \ g(T) = \mu(W(T)). \)

This decomposition is more 'complicated' than Exs. 1, 2 & 3. Computation of \( f \) and \( g \) are also significantly more difficult — but doable.

\[ \Rightarrow \text{Luo-T identity} \]
References:


Other relevant papers by Basmajian (AJM), McShane (Inventiones), Bridgeman (G & T), Bridgeman-Kahn (G AFA), Mirzakhani (Inventiones), T-Wong-Zhang (JDG), Huang-Norbury (G I), Vlamis-Yarmola (to appear J Top), Labouvie-McShane (Duke), …..
Thank You 谢谢.