Cable surfaces whose automorphism groups are virtually free

S. Mukai
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IMS @ Singapore

\( S : \) Enriques/Coble surface
\( \text{Aut } S : \) discrete group
( subgp of the plane Cremona group \( C_{12} \)
when Cable)

\[ \text{Conjecture} \]
\[ \text{vcd} \left( \text{Aut } S \right) = \max \text{MW-rank} \left( f \right) \]
\[ f: S \rightarrow \mathbb{P}^4 \]

\( \text{genus one fibration} \)

§1. Cable surface with irreducible boundary

\( p_{i} \in C \subset \mathbb{P}^2 \)

\( \uparrow \uparrow \)

\( B \subset Bl_{10} \mathbb{P}^2 \)

plane sextic with 10 nodes at \( p_{i,1} \ldots p_{i,10} \)

Pair \( S \) blow-up of \( \mathbb{P}^2 \) at \( p_{i,1} \ldots p_{i,10} \)
and the strict transform \( B \),

is a Cable surface in the following sense.
(Since \( g(B) = \chi_p(C) - 10 = 0 \), \( B \cong \mathbb{P}^1 \)).

**Definition** \( X : \) K3 surface, \( \epsilon \) \( \mathbb{R} \times \) involution such that \( \text{Fix } \epsilon \) is a disjoint union of \( m \) \( \mathbb{P}^1 \)'s. Quotient and the images of \( \mathbb{P}^1 \)'s \( (X/\epsilon, \mathbb{P}^1) \) is called an Enriques surface when \( m = 0 \) and a Coble surface (with \( m \) boundary components) when \( m > 0 \).

In our case, the pair \((\text{Bl}_{10} \mathbb{P}^2, B)\) is a Coble surface since the double cover \( X \rightarrow \text{Bl}_{10} \mathbb{P}^2 \) with branch \( B \) is a K3 surface and \( B \cong \mathbb{P}^1 \).

\[
\text{Pic } B = H^2(S, \mathbb{Z}) \cong \mathbb{Z} \oplus \bigoplus_{i=1}^{10} \mathbb{Z} \cdot e_i
\]

where \( e_i \) is the class of exceptional divisor. C

\[
\left[ \begin{array}{c}
P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \\ P_6 \\ P_7 \\ P_8 \\ P_9 \\ P_{10}
\end{array} \right] \rightarrow 2
\]
(1, e_1, ..., e_n) orthonormal basis with \((e_i^2) = 1\) and \((e_i^2) = -1\) for \(i \neq 2\).

\[ \begin{bmatrix} 3 & x - \frac{1}{x} e_i \end{bmatrix} \] unique member and 
\[ c_1(B_{l_{10}, P^2}) = 3 - \frac{1}{x} e_2. \] Hence, \(B\) is preserved by \(\text{Aut}(B_{l_{10}, P^2})\).

Orthogonal complement \([B]^{\perp}\) in \(H^2(C, \mathbb{Z})\)
is \(E_{10} = T_2, 3, 7\) even unimodular lattice of signature \((1, 9)\), with standard basis:

\[
e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9, e_{10}
\]

\(\text{th} - e_1 - e_2 - e_3 \quad \begin{cases} (x^2) = -2 \\ \alpha \beta = 1 \\ \alpha \gamma = 0 \end{cases}\)

\((C_0)\) homological representation

\[ \text{Aut} S \longrightarrow O_{\mathbb{Z}}(E_{10}) \]

(Coleb, Barili) If \(C\) is generic, then \(S \not\subset B\) does not contain \(P^1\) and
8.2 Virtual cohomological dimension (v.c.d.)

\[ \Gamma : \text{discrete group} \]
\[ \text{vcd} \, (\Gamma) = \sup \{ n \mid H^n(\Gamma, \mathbb{Z}) \neq 0 \} \]
\[ \text{vcd} \, \mathbb{Z}^n = n \quad \text{free abelian group} \]
\[ \text{vcd} \, F_n = 1 \quad \text{free group} \]

Free action on contractible space
\[ \mathbb{Z}^n \not\rightarrow \mathbb{R}^n, \quad F_n \not\rightarrow \text{true} \]
\[ \text{vcd} \, \Gamma = 0 \iff \text{finite group} \]
\[ \text{vcd} \, \Gamma = 1 \iff \text{virtually free, i.e., } \exists \text{ free subgroup } \]
\[ \Gamma_0 \subset \Gamma \text{ of finite index} \]

**Example:** \( \text{O}_2(\mathbb{E}_{10}) \) acts on the 9-dimensional Lobachevsky space
\[ \{ (x^2) < 0 \} \subset \mathcal{P}(\mathbb{E}_{10} \otimes \mathbb{R}) \text{ virtually freely} \]

\[ \text{vcd} \, \text{O}_2(\mathbb{E}_{10}) = 9 - (\mathbb{Q}-\text{rank}) = 8 \]

\[ \text{Borel-Serre bordification} \]
§ 3  Modoell-Weil (MW) rank and

Enriques/Coble surface

\[ \begin{array}{c}
\text{MW} \circ \text{rank} \circ (\text{Jac} \circ f) = \text{MW} \circ (\text{Jac} \circ f) = \text{MW} \circ (\text{Jac} \circ f) \\
\end{array} \]

Jacobian fibration is a rational elliptic surface, obtained from \( \mathbb{P}^2 \) by blowing up 9 times.

\[ \text{MW} \circ \text{rank} \circ (\text{Jac} \circ f) = \text{MW} \circ (\text{Jac} \circ f) = \text{MW} \circ (\text{Jac} \circ f) \]

(\text{group of sections of})

\[ \text{Jac} \circ f \]

is finitely generated abelian group, the MW group. This acts on \( \mathcal{S} \) by translation.

Shioda-Tate formula

\[ \text{MW-rank} \circ (\text{Jac} \circ f) = 8 - \sum_{\text{fibre}} \left( \text{# of \text{invol. comp.}} \right) - 1 \]

\[ \text{MW-rank} \circ (\text{f}) = 8 - \sum_{\text{fibre}} \left( \text{# of \text{invol. comp.}} \right) - 1 \]

Example 2 (ii) general case

\[ R = \text{Bl}_g \mathbb{P}^2 \to \mathbb{P}^2 \]

\[ \to 5 \]
Blow-up at the $g$ base points of a general

period $4$ plane cubics $<C_1, C_2>$,

$\rightarrow$ All fibers are reducible and MW-rank $= 8$

($\Leftrightarrow \exists \mathbb{P}^1$ in fibers)

(2) **Pappus** $(9,3)$ configuration

$R = \text{Bl}_g \mathbb{P}^2 \rightarrow \mathbb{P}^2$

contains $9$ $\mathbb{P}^1$'s in fibers:

They make $3$ triangles.

$\text{MW-rank} \leq 8 - (2 + 2 + 2) = 2$

("=" holds up the configuration is general.)

$\text{Bl}_g \mathbb{P}^2$

$\begin{array}{c}
|3+2\mathbb{P}^1| \\
\downarrow \\
\mathbb{P}^1
\end{array}$
§4 Two extremes where the conjecture holds

(1) Rational plane sextic $C \subset \mathbb{P}^2$ is very general

$$(\text{LHS}) = \gcd O \mathbb{Z} (E_0)(E) = \delta \quad \text{by Galois-Barth}$$

$$(\text{RHS}) = \delta \quad \text{by Struza-Tate formula}$$

since $S \not\subset B$ does not contain $\delta$.

(2) If $(\text{RHS}) = 0$, then essentially by

Vinbarg criterion $Aut S$ is finite. Hence

$(\text{LHS}) = 0$. (In 190's, such Enriques surfaces were classified by Nitsaulin-Kondo.)

§5 Cable surface with $\gcd (Aut S) = 1$

Wiman sextic

$$W: x^6 + y^6 + z^6 + \left(x^2 y^2 z^2 + x^2 y^3 + y^3 z^2\right) + 12 x^2 y^2 z^2$$

has 16 nodes at

- $(\pm 1: \pm 1: 1)$ and
- $(0: 1: 2), (+2: 0: 1), (1: 2: 0)$
where \( \mathcal{C} = (1 + \sqrt{5}) / 2 \). Let \( S = \text{Bl}_{x_0} \mathbb{P}^2 \) be the blow-up at the 10 nodes and \( \mathcal{B} \) the strict transform of \( x_0 \in W \).

**Theorem. (LHS)**

\[
\text{Aut } S = \mathcal{U}_5 + \mathcal{U}_4
\]

\[
1 \rightarrow F_9 \rightarrow \text{Aut } S \rightarrow \mathcal{G}_3 \rightarrow 1 \quad \text{(exact)}
\]

In particular, \( \text{Aut } S \) is virtually free.

**RHS** The interior \( S \setminus \mathcal{B} \) contains 15 \( \mathcal{P}^1 \)s, \( \{ C_i \}_{i=1}^3 \), with Petersen configuration.

\[
C_i \cong \mathcal{P}^1 \quad \Longleftrightarrow \quad \text{edge of } \mathcal{P}_{10}
\]

\[
(C_i, C_j) = \begin{cases} 1 & \text{common vertex} \\ 0 & \text{otherwise} \end{cases}
\]

Modulo the action by \( F_9 \), all genera one fibrations of \( S \) are classified and we obtain

\[
\text{RHS} = 1
\]

**Corollary.** The conjecture holds in the case of Wiman sextic. (LHS = 1 = RHS)
Idea of Proof of (LHS) Similar to that

\[ \text{SL}(2, \mathbb{Z}) \cong C_4 \ast C_2 \quad (C_n: \text{cyclic of order } n) \]

\[ \text{SL}(2, \mathbb{Z}) \curvearrowright \text{upper } \frac{1}{2} \text{ plane} \]

virtually free action on
tree with stabilizer groups
\( C_4, C_6 \) and \( C_2 \).

\( \text{Aut} \ S \cong \text{Stab}_{[A]} \ast \text{Stab}_{[B]} \)

dual cone
of \( \text{Eff}(S) \)

\( \exists \) true-like chamber decomposition

\[ S = \text{Bl}_{10} \text{IP}^2 \]

\[ \text{Bl}_{4A} \text{IP}^2 = \text{R}_{S} \]

\[ \text{Bl}_{6B} \text{IP}^2 = \text{R}_{S} \quad \text{Clebsch cubic with } G_{5}-\text{symmetry} \]

\[ \text{Stab}_{[A]} = \text{Aut} (\text{R}_{S}, 4 \text{A}) \cong G_{4} \]

\[ \text{Stab}_{[B]} = \text{Aut} (\text{R}_{S}, 4 \text{B}) \cong U(5) \]