Building Blocks of Polarized Endomorphisms of Normal Projective Varieties

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Let $X$ be a (normal) projective variety.
Let $f : X \to X$ be a surjective endomorphism.
Denote by $N^1(X) := \text{NS}(X) \otimes_{\mathbb{Z}} \mathbb{R}$ for the Néron-Severi group $\text{NS}(X) = \text{Pic}(X) / \text{Pic}^0(X)$.
$f$ induces an automorphism $f^* : N^1(X) \to N^1(X)$. 
Preliminary

Definition (Polarized Endomorphism)

Let $f : X \to X$ be a surjective endomorphism of a (normal) projective variety $X$. We say that $f$ is polarized if there is an ample Cartier divisor $H$ such that $f^*H \sim qH$ for some integer $q > 1$.

Proposition

The following are equivalent.

1. $f$ is polarized.
2. $f^*H \equiv qH$ in $N^1(X)$ for some ample $\mathbb{R}$-divisor $H$.
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$X$ is called $Q$-abelian if there is a quasi-étale (i.e. étale in codimension 1) cover $\pi : A \to X$ from an abelian variety $A$. 

Theorem (N. Nakayama and D. Q. Zhang, 2010) Let $X$ be a normal projective surface and $f$ a polarized endomorphism. Then either $f^*|_{N^1(X)}$ is a scalar (after iteration) or $X$ is $Q$-abelian.

Question: what about high dimensional case?
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Let $X$ be $\mathbb{Q}$-factorial lc and $f : X \to X$ be a polarized endomorphism. Then any MMP (with finitely many steps) starting from $X$ can be run $f$-equivariantly (after iteration).
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Sketch of the idea: Let $f_i = f|_{X_i}$ and $\pi : X_i \to X_{i+1}$ the $i$th step. We need to show (1) $f_i$ may descend to an endomorphism $f_{i+1} : X_{i+1} \to X_{i+1}$; (2) $f_{i+1}$ is again polarized.
Main Results

For (1), the key observation is the following:

**Lemma**

Let $(X, f)$ be a polarized pair. Suppose $A \subseteq X$ is a closed subvariety with $f^{-i}f^i(A) = A$ for all $i \geq 0$. Then $f^\pm(A) = A$ (after iteration).
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**Lemma**

Let \((X, f)\) be a polarized pair. Suppose \(A \subseteq X\) is a closed subvariety with \(f^{-i}f^i(A) = A\) for all \(i \geq 0\). Then \(f^{\pm 1}(A) = A\) (after iteration).

**Lemma**

Let \(X\) be a \(\mathbb{Q}\)-factorial lc normal projective variety and \(f : X \to X\) a surjective endomorphism. Let \(\pi : X \to Y\) be a contraction of a \(K_X\)-negative extremal ray \(R_C := \mathbb{R}_{\geq 0}[C]\). Suppose that \(E \subseteq X\) is a subvariety such that \(\dim(\pi(E)) < \dim(E)\) and \(f^{-1}(E) = E\). Then replacing \(f\) by a positive power, \(f(R_C) = R_C\); hence, \(\pi\) is \(f\)-equivariant.
For (2), first note that by pull back of $\pi_i$, $N^1(X_{i+1})$ can be regarded as an $((f^*)^\pm 1$-invariant) hyperplane of $N^1(X_i)$ (divisoral and Fano contractions) or just $N^1(X_i)$ (flip).

Remark 1. $f^*i$ is a scalar iff so is $f^*i + 1$.

2. The ample cones $\text{Amp}(X_{i+1}) \cap \text{Amp}(X_i) = \emptyset$. 
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**Remark**

1. $f_i^*$ is a scalar iff so is $f_{i+1}^*$.
2. The ample cones $\text{Amp}(X_{i+1}) \cap \text{Amp}(X_i) = \emptyset$. 
Let $f : V \to V$ be a (linear) automorphism of a positive dimensional real vector space $V$ such that $f^{\pm 1}(C) = C$ for a closed cone $C \subseteq V$ which spans $V$ and contains no line. Let $q$ be a positive number. Then (1) and (2) below are equivalent.

(1) $f(x) = qx$ for some $x \in C^\circ$.

(2) There exists a constant $N > 0$, such that $\frac{\|f^i\|}{q^i} < N$ for any $i \in \mathbb{Z}$.

If (1) or (2) above is true, then $f$ is a diagonalizable linear map with all eigenvalues of modulus $q$. 
Application: $C = \text{Nef}(X) \coloneqq \overline{\text{Amp}(X)}$ or $C = \text{PEC}(X) \coloneqq \overline{\text{Big}(X)}$

**Proposition (S. Meng and D.-Q. Zhang)**

Below are equivalent.

1. $f$ is polarized.
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Proposition (S. Meng and D.-Q. Zhang)

Let \( \pi : X \dashrightarrow Y \) be a dominant rational map between two projective varieties and let \( f : X \to X \) and \( g : Y \to Y \) be two surjective endomorphisms such that \( g \circ \pi = \pi \circ f \). If \( f \) is polarized. Then \( g \) is polarized.
Non-uniruled Case

Theorem (D. Greb, S. Kebekus and T. Peternell, 2016)

Let $X$ be a non-uniruled normal projective variety and $f$ polarized. Then $X$ is $Q$-abelian.

As an application, we have:

Lemma (S. Meng and D.-Q. Zhang)

Let $(X, f)$ be a polarized pair with $X$ being a $Q$-factorial klt normal projective variety. Assume that $K_X$ is pseudo-effective. Then $X$ is $Q$-abelian.

Remark

Let $X$ be a $Q$-abelian variety and $f: X \to X$ a surjective endomorphism. Assume the existence of a non-empty closed subset $Z \subset X$ and $s > 0$, such that $f^{-s}(Z) = Z$. Then $f$ is not polarized.
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Theorem (S. Meng and D.-Q. Zhang)

Let \((X, f)\) be a polarized pair such that \(X\) has at worst \(\mathbb{Q}\)-factorial klt singularities. Then, replacing \(f\) by a positive power, there exist a \(\mathbb{Q}\)-abelian variety \(Y\), a morphism \(X \to Y\), and an \(f\)-equivariant relative MMP over \(Y\) such that we have:

\[
X = X_1 \dasharrow \cdots \dasharrow X_i \dasharrow \cdots \dasharrow X_r = Y
\]

(i.e. \(f = f_1\) descends to polarized \(f_i\) on each \(X_i\)), such that we have:

1. If \(K_X\) is pseudo-effective, then \(X = Y\) and it is \(\mathbb{Q}\)-abelian.
2. If \(K_X\) is not pseudo-effective, then for each \(i\), \(X_i \to Y\) is an equi-dimensional morphism with every fibre (irreducible) rationally connected. The \(X_{r-1} \to X_r = Y\) is a Fano contraction.
Theorem (S. Meng and D.-Q. Zhang)

Let \((X, f)\) be a polarized pair. Assume that \(X\) is smooth and rationally connected. Then, replacing \(f\) by a positive power, \(f^*\) is a scalar.

Sketch:
\(X\) has no non-trivial (quasi)-étale cover. So MMP ends up with a point.

Question
Let \((X, f)\) be a polarized pair. Assume that \(X\) is a rationally connected variety with at worst \(Q\)-factorial terminal singularities. Is a positive power of \(f\) a scalar? True when \(\text{dim}(X) \leq 3\).
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Main Results (Application)

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Thanks!