Weil-Petersson metric and hyperbolicity problems of some families of polarized manifolds

Conference on Complex Geometry, Dynamic Systems and Foliation Theory
Institute for Mathematical Sciences
National University of Singapore
May 15-19, 2017

Sai-Kee Yeung
Purdue University

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Outline of the talk

I. Introduction

II. Results

III. Idea of proofs
   (i). Hyperbolicity of moduli
   (ii). Log-Kodaira dimension of moduli
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I. Introduction

Joint work with Wing-Keung To.

In complex geometry, we can measure negativity, or hyperbolicity, of a Kähler manifold \((\mathcal{M}, g)\), in terms of curvature of a Kähler metric \(g\), such as

- (i) Holomorphic sectional curvature:
  \[ R_{\alpha \beta \alpha \alpha} < 0, \mid \alpha \mid = 1; \]
  where
  \[ R_{\alpha \beta \gamma \delta} = -\partial_a \partial_{\beta} g_{\gamma \delta} + g_{\mu \bar{\nu}} \partial_\alpha \bar{\nu} \partial_{\mu} \gamma. \]

- (ii) Ricci curvature,
  \[ R_{\alpha \beta} = g_{\mu \bar{\nu}} R_{\alpha \beta \mu \bar{\nu}} < 0. \]

Can also describe in a more (holomorphically) invariant way:

- (iii) Complex hyperbolicity, such as Kobayashi hyperbolic, or
- (iv) General type, or Log-general type properties,
  \[ \dim \Gamma(\mathcal{M}, aK_{\mathcal{M}}) \geq c_n \]
  i.e. \( \kappa(K_{\mathcal{M}}) = n \), or
  \[ \dim \Gamma(\mathcal{M}, a(K_{\mathcal{M}} + D)) \geq c_n \]
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In this talk, we primarily focus on \( \chi : S \to \mathbb{C} \)

\( \) is a family of complex manifolds over a base \( S \),

where fiber \( M_s \) is a complex manifold and \( S \) is also a complex manifold.

Consider first \( M \) is a Riemann surface with \( \dim \mathbb{C} M = 1 \):

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\( g \)\((M)\) \( \geq 2 \)

\( \mathbb{C} \)/\( \mathbb{SL}(2, \mathbb{Z}) \)

\( \dim \mathbb{C} : 3 \)

\( g - 3 \)

Moduli \( M_g \) parametrize isomorphism classes of curves of genus \( g \).

Similarly, we may consider \( M_g, n \), moduli of Riemann surfaces of genus \( g \) with \( n \) punctures.

\( M_g, M_g, n \) for \( g \geq 2 \) share the following properties:

- negatively curved,
- hyperbolic,
- and are of log-general type.
In this talk, we primarily focus on \( \pi : \chi \to S \) a family of complex manifolds over a base \( S \).
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$\frac{\tilde{M}}{P^1_{\mathbb{C}}, \mathbb{C}} \cong \frac{M}{\Gamma}$ 

$\{\cdot\} \subset \text{H}/SL(2, \mathbb{Z})$

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Moduli $M_g$: parametrize isomorphism classes of curves of genus $g$.
Similarly, we may consider $M_{g,n}$, moduli of Riemann surfaces of genus $g$ with $n$ punctures.

$M_g, M_{g,n}$ for $g \geq 2$ share the following properties: 
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<table>
<thead>
<tr>
<th>$\tilde{M}$</th>
<th>$\mathbb{P}^1_\mathbb{C}$</th>
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</tr>
</thead>
<tbody>
<tr>
<td>$g(M)$</td>
<td>0</td>
<td>1</td>
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<tr>
<td>$\mathcal{M}_g$</td>
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\[
\begin{array}{|c|c|c|c|}
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\tilde{M} & P^1_{\mathbb{C}} & \mathbb{C} & \Delta \\
\hline
g(M) & 0 & 1 & \geq 2 \\
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Recall some standard terminology.

Kobayashi infinitesimal pseudo-metric:
\[ \sqrt{g_K}(x, v) := \inf \{ 1 \mathbb{R} \mid \exists f : \Delta \mathbb{R} \rightarrow M_{hol}, f(0) = x, f'(0) = v \} \]

Kobayashi distance:
\[ d_K(x, y) = \inf \{ \ell \mid \text{joining } x \text{ and } y \} \]

\( M \) is Kobayashi hyperbolic:
\[ d_K(x, y) > 0 \quad \forall x \neq y. \]

\( M \) is Brody hyperbolic:
\( \nexists f : \mathbb{C} \rightarrow M \) non-constant.

Kobayashi hyperbolic \( \Rightarrow \) Brody hyperbolic

Kobayashi hyperbolic \( \Leftarrow \) Brody hyperbolic if \( M \) compact (Brody Re-parametrization)

In general \( \nexists \) if \( M \) non-compact
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An easy criterion for hyperbolicity: $(M, g)$ has holomorphic sectional curvature $\leq c < 0 \Rightarrow M$ Kobayashi hyperbolic.

Reason:

(i) Recall Ahlfors' Schwarz Lemma: For $f: \Delta \rightarrow M$ holomorphic, $f^* g_{\Delta} \leq 1$.

Poincaré metric $g_{\Delta} = \frac{1}{R^2} |dz|^2 (R^2 - |z|^2)^2$.

At $z = 0$, $g_{\Delta}(0) = |dz|^2 R^2$.

(ii) Apply Lemma to $f: \Delta \rightarrow M$, at 0, with $df (\frac{\partial}{\partial z}) = v$, $\Rightarrow R$ is bounded above $\Rightarrow |v| g_K > 0$. 
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eg 1. Riemann surface, genus $\geq 2$, is hyperbolic, since universal cover $= \Delta$, equipped with $g_\Delta$.

eg 2. $\mathbb{P}C \{0, 1, \infty\}$ is hyperbolic, since universal cover $= \mathbb{H}$, note: $\mathbb{P}C \{0, 1, \infty\} = \mathbb{H} / \left[ \text{SL}_2(\mathbb{Z}) \right]$.

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P_C \setminus \{0, 1, \infty\} = \mathcal{H}/[SL_2(\mathbb{Z}), SL_2(\mathbb{Z})]
\]

\[
\downarrow
\]

\[
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Let $t \in \mathcal{M}_g$. $t$ represents a Riemann surface $M_t$ of genus $g$. 

Ahlfors (61), Royden (75), Wolpert (86): holomorphic sectional curvature $R_{\alpha\beta\gamma\delta} \leq -\frac{1}{2} \pi (g - 1)$.

In particular, $M_g$ is Kobayashi hyperbolic if $g \geq 2$. 

I. Introduction

- Let $t \in \mathcal{M}_g$. $t$ represent a Riemann surface $M_t$ of genus $g$.
  - $\exists$ a natural invariant metric on $\mathcal{M}_g$: Weil-Petersson $g_{WP}$. 

$R_{\alpha\beta\gamma\delta} = -2 \int_{M_t} \left( (\Box + 2)^{-1} \langle \Phi_\alpha, \Phi_\beta \rangle \right) \cdot \langle \Phi_\gamma, \Phi_\delta \rangle \omega - 2 \int_{M_t} \left( (\Box + 2)^{-1} \langle \Phi_\alpha, \Phi_\delta \rangle \right) \cdot \langle \Phi_\gamma, \Phi_\beta \rangle \omega$

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- It is known classically (or from Kodaira-Spencer) that tangent vectors to the moduli at point $t$ are determined by harmonic $\Phi \in H^1(M_t, T_{M_t})$.
- Define $g_{WP}(v_1, \bar{v}_2) := \int_{M_t} \langle v_1, \bar{v}_2 \rangle_{g_{\Delta}} dv_{g_{\Delta}}$. 

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Let $t \in \mathcal{M}_g$. $t$ represent a Riemann surface $M_t$ of genus $g$. There exists a natural invariant metric on $\mathcal{M}_g$: Weil-Petersson $g_{WP}$. It is known classically (or from Kodaira-Spencer) that tangent vectors to the moduli at point $t$ are determined by harmonic $\Phi \in H^1(M_t, T_{M_t})$. Define $g_{WP}(v_1, \bar{v}_2) := \int_{M_t} \langle v_1, \bar{v}_2 \rangle_{g_\Delta} dv_{g_\Delta}$.

\[ R_{\alpha \beta \gamma \delta} = -2 \int_{M_t} ((\Box + 2)^{-1} \langle \Phi_\alpha, \Phi_\beta \rangle) \cdot \langle \Phi_\gamma, \Phi_\delta \rangle \omega \]
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holomorphic sectional curvature $R_{\alpha\overline{\alpha}\alpha\overline{\alpha}} \leq -\frac{1}{2\pi(g-1)}$,

In particular, $\mathcal{M}_g$ is Kobayashi hyperbolic if $g \geq 2$. 
The goal here is to generalize the results to family of higher dimensional varieties of the following three types of manifolds on the fiber.

(a) Family of canonically polarized manifolds i.e. $K^M$ ample, or Kähler-Einstein with negative scalar curvature, $R_{ij} = g_{ij}, c < 0$.

(b) Family of polarized Ricci flat Kähler manifolds and orbifolds. i.e. Ricci curvature $R_{ij} = 0$.

(c) Family of log-canonically polarized manifolds, i.e. $M$ equipped with complete Kähler-Einstein metrics.

We are going to prove that some (augmented) Weil-Petersson metric

(a) possess a Finsler metric with $R^{\alpha\alpha} \alpha \leq c < 0$, hence is Kobayashi hyperbolic;

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II. Results

Consider (a) moduli space of Kähler-Einstein metric of negative scalar curvature (⇔ canonically polarized manifolds).

(I) (Migliorino, Kovacs, Kebekus-Kovacs,...) Given a family of canonically polarized manifolds over an algebraic curve $C$, $g(C) = 0 \Rightarrow \exists \geq 3$ singular fibers, $g(C) = 1 \Rightarrow \exists \geq 1$ singular fiber.

(II) (Zuo-Viehweg 2003) Let $\pi: X \rightarrow S$ be an effectively parametrized holomorphic family of K.E. manifolds (-ve curv) over a complex manifold $S$. Then $S$ is Brody hyperbolic.

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Finsler metric: length function \( h \) on \( T \) \( M \) satisfying \( |cv|_h = |c_v|_h \).

Effectively parametrization: Kodaira-Spencer map \( \rho \) : \( T^\infty S \to H^1(M_t, T^\infty M_t) \) is injective.

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Schumacher independently constructed a Finsler metric of negative hol. sect. curvature, but no upper bound $\leq -c < 0$, cannot conclude hyperbolicity directly.

Proofs of theorems of Viehweg-Zuo, Migliorino, Kovacs, Kebecus-Kovacs etc. are algebraic in nature.

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II. Results

Consider the family of Kähler Ricci-flat manifolds or orbifolds. One case corresponds to moduli of elliptic curves.

Theorem (To-Yeung (b))

Let \( \pi : X \to S \) be an effectively parametrized holomorphic family of compact polarized Kähler manifolds of zero first Chern class over a complex manifold \( S \). Then \( S \) admits a \( C^\infty \) \( \text{Aut}(\pi) \)-invariant Finsler metric, with holomorphic sectional curvature \( \leq -c < 0 \), where \( c \) is a constant. Hence \( S \) is Kobayashi hyperbolic.

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Same conclusion for family of compact polarized Ricci-flat Kähler orbifolds.
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Same conclusion for family of compact polarized Ricci-flat Kähler orbifolds.
II. Results

Consider the family of quasi-projective manifolds $M = M - D$ satisfying:

(i) $D = \sum_{i=1}^{l} D_i$, with $D_i$ simple normal crossing,

(ii) $(K_M + D_i)|_{D_i} > 0 \forall i$.

It follows that $M$ is equipped with a complete K"ahler-Einstein metric $g$ of negative scalar curvature with bounded geometry, i.e.:

(i) The curvature tensor is bounded on $M$,

(ii) The volume of $(M, g)$ is finite.

(Tsuji, Tian-Yau, Wu, ...)

For this article, we call $M$ 'log-canonically polarized'.
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Example:

Let $M_t$ be a family of smooth hyperplanes in $\mathbb{P}^n_{\mathbb{C}}$ of large degree. Let $H$ be a smooth hypersurface of sufficiently large degree in $\mathbb{P}^n_{\mathbb{C}}$. Let $D_t = H \cap M_t$, defined by 

$$s_t$$

as a divisor on $M_t$. Let $S$ be the set of $t$ such that the intersection $H \cap M_t$ is transversal. 

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Let $\pi : X \to S$ be an effectively parametrized holomorphic family of log-canonically polarized manifolds with bounded variation over a complex manifold $S$. Then $S$ admits a $C^\infty$ Aut($\pi$)-inv Finsler metric, with holomorphic sectional curvature $\leq -c < 0$, where $c$ is a constant. Hence $S$ is Kobayashi hyperbolic.
II. Results

We study another type of hyperbolicity criterion. Getting back to (a), family of (canonically) polarized manifolds.

Conjecture (Viehweg)
Let \( \pi : \chi \to S \) be an effectively parametrized family of canonically polarized manifolds. Assume that \( S = S - D \), \( D \) simple normal crossing divisor. Then \( S \) is of log-general type, i.e. \( K_S + D \) is big.

Results for canonically polarized ones:
(a). \( \text{dim} = 1 \): Shafarevich Conjecture, solved by Parshin, Arakelov.
(b). Arbitrary dimension: partial results were obtained by Kebekus-Kovacs (dim 3), Patakfalvi (S compact), Campana-Paun (general).

For (b), all depends on existence of a Viehweg-Zuo subsheaf.

(Viehweg-Zuo) There exists a big subsheaf \( F \) of \( \otimes m \Omega(\chi, D) \) for some \( m \in \mathbb{Z}^+ \) (for canonically polarized family).
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    (a). \( \dim = 1 \): Shafarevich Conjecture, solved by Parshin, Arakelov.
II. Results

- We study another type hyperbolicity criterion. Getting back to (a), family of (canonically) polarized manifolds.

- **Conjecture (Viehweg)**

  Let $\pi : \chi \to S$ be an effectively parametrized family of canonically polarized manifolds. Assume that $S = \bar{S} - D$, $D$ simple normal crossing divisor. Then $S$ is of log-general type, i.e. $K_{\bar{S}} + D$ is big.

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- For (b), all depends on existence of a Viehweg-Zuo subsheaf.

- **(Viehweg-Zuo)** There exists a big subsheaf \(\mathcal{F}\) of \(\otimes^m \Omega(S, D)\) for some \(m \in \mathbb{Z}^+\) (for canonically polarized family).
II. Results

We give a direct construction of a sheaf of Viehweg-Zuo type for the case of (a), (b) and (c) and derive log-general properties as desired.

Theorem (To-Yeung)

Let $\pi: \chi \to S$ be an effectively parametrized family of manifolds which are one of the following types

(a) canonically polarized,
(b) log-polarized Kähler-Ricci flat,
(c) log-canonically polarized.

Assume that $S = S - D$, where $D$ is a simple normal crossing divisor. Then

(i). There exists explicitly a Viehweg-Zuo subsheaf of $\otimes^m \Omega(S, D)$ for some $m$.
(ii). $S$ is of log-general type.
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III. Idea of proof of (a)
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Consider a family

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\( R_{a\bar{\beta}}(t) = kg_{a\bar{\beta}}(t), \ k < 0 \)

\( \omega_{\mathcal{M}} = \frac{2\pi}{k} c_1(K^{-1}_{X|S}, g). \)
Given a local tangent vector field $u$ on $S$, there is a unique lifting to $\nu_u$ such that $\pi_*(\nu_u) = u$. 

Hence Kodaira-Spencer Map $\rho_t: T_t S \to H^1(M_t, T_{M_t})$ is represented by $\Phi(u(t))$, a harmonic bundle-valued form on $M_t$. 
Given a local tangent vector field $u$ on $S$, there is a unique lifting to $v_u$ such that such that $\pi_*(v_u) = u$. 
$\Phi(u(t)) := \bar{\partial} v_u|_{M_t} \in A^{0,1}(M_t)$ is actually a harmonic representative of the Kodaira-Spencer class, called canonical lift (Siu) or horizontal lift (Schumacher).
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Hence Kodaira-Spencer Map $\rho_t : T_t S \rightarrow H^1(M_t, T_{M_t})$ is represented by $\Phi(u(t))$, a $\square_t = \partial\partial^* + \partial^*\partial$ harmonic bundle-valued form on $M_t$. 
III. Idea of proof of (a)

For \( v_i \in T \), denote \( \Phi_i = \rho(v_i) \).

Define \( h_{WP_{ij}} = \int_M \langle \Phi_i, \Phi_j \rangle \omega \) w.r.t. normal coordinates.

Note that we are using a 'canonical' or 'horizontal' lifting of \( v \) to total space.
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- (Siu 86, Schumacher 93)

$$R_{ijk\ell}^{(WP)}(t) = k \int_{M_t} ((\Box - k)^{-1} \langle \Phi_i, \Phi_j \rangle) \cdot \langle \Phi_k, \Phi_\ell \rangle \frac{\omega^n}{n!}$$

$$+ k \int_{M_t} ((\Box - k)^{-1} \langle \Phi_k, \Phi_j \rangle) \cdot \langle \Phi_i, \Phi_\ell \rangle \frac{\omega^n}{n!}$$

$$+ k \int_{M_t} \langle (\Box - k)^{-1} \mathcal{L}_{v_i} \Phi_k, \mathcal{L}_{v_j} \Phi_\ell \rangle \frac{\omega^n}{n!}$$

$$+ \int_{M_t} \langle H(\Phi_i \otimes \Phi_k), H(\Phi_j \otimes \Phi_\ell) \rangle \frac{\omega^n}{n!}.$$ 

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\[
R^{(WP)}_{i j k \ell}(t) = k \int_{M_t} \left( (\square - k)^{-1} \langle \Phi_i, \Phi_j \rangle \right) \cdot \langle \Phi_k, \Phi_\ell \rangle \frac{\omega^n}{n!} \\
+ k \int_{M_t} \left( (\square - k)^{-1} \langle \Phi_k, \Phi_j \rangle \right) \cdot \langle \Phi_i, \Phi_\ell \rangle \frac{\omega^n}{n!} \\
+ k \int_{M_t} \langle (\square - k)^{-1} \mathcal{L}_{v_i} \Phi_k, \mathcal{L}_{v_j} \Phi_\ell \rangle \frac{\omega^n}{n!} \\
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- Procedures to obtain the above identity:
  Let $\Psi \in \mathcal{A}^{0,1}(M_t, T_{M_t})$, representing $v \in T_tS$. Let $\frac{\partial}{\partial t_i} \in T_tS$. 

\[\frac{\partial}{\partial t_i} \parallel \Psi \parallel^2_2 = \frac{\partial}{\partial t_i} \left( \frac{\partial}{\partial t_i} \parallel \Psi \parallel^2_2 \parallel \Psi \parallel^2_2 \right) = \frac{\partial}{\partial t_i} \parallel \Psi \parallel^2_2 - \left( \frac{\partial}{\partial t_i} \parallel \Psi \parallel^2_2 \right) \left( \frac{\partial}{\partial t_i} \parallel \Psi \parallel^2_2 \right) \parallel \Psi \parallel^4_2.\]

\[\frac{\partial}{\partial t_i} \parallel \Psi \parallel^2_2 = \frac{\partial}{\partial t_i} \int_{M_t} \langle \Psi, \Psi \rangle \omega^n = \int_{M_t} \langle L v_i \Psi, \Psi \rangle \omega^n + \int_{M_t} \langle \Psi, L v_i \Psi \rangle \omega^n = \int_{M_t} \langle L v_i \Psi, \Psi \rangle \omega^n.\]
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\[
\partial_i \partial_i \log \|\Psi\|_2^2 = \partial_i \left( \frac{\partial_i \|\Psi\|_2^2}{\|\Psi\|_2^2} \right) = \frac{\partial_i \partial_i \|\Psi\|_2^2}{\|\Psi\|_2^2} - \frac{(\partial_i \|\Psi\|_2^2)(\partial_i \|\Psi\|_2^2)}{\|\Psi\|_4^2}.
\]
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► Procedures to obtain the above identity:

Let $\Psi \in \mathcal{A}^{0,1}(M_t, T_{M_t})$, representing $v \in T_tS$. Let $\frac{\partial}{\partial t_i} \in T_tS$.

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\]

\[
\partial_i \|\Psi\|_2^2 = \frac{\partial}{\partial t_i} \int_{M_t} \langle \Psi, \Psi \rangle \frac{\omega^n}{n!} \\
= \int_{M_t} \langle \mathcal{L}_{v_i} \Psi, \Psi \rangle \frac{\omega^n}{n!} + \int_{M_t} \langle \Psi, \mathcal{L}_{v_i} \Psi \rangle \frac{\omega^n}{n!} \\
= \int_{M_t} \langle \mathcal{L}_{v_i} \Psi, \Psi \rangle \frac{\omega^n}{n!}. 
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III. Idea of proof of (a)

Key point: To handle each terms by integration by part guided by geometry.

Obvious strategy: Control the last term by the others.

(People tried for years.)
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\[
\partial_i \partial_i \|\psi\|^2_2 = \partial_i \partial_i \|\psi\|^2_2 = \frac{\partial}{\partial t^i} \int_{M_t} \langle \mathcal{L}_v \psi, \psi \rangle \frac{\omega^n}{n!}
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\[ \partial_i \partial_i \| \Psi \|_2^2 = \partial_i \partial_i \| \Psi \|_2^2 = \frac{\partial}{\partial t} \int_{M_t} \langle L_{v_i} \Psi, \Psi \rangle \frac{\omega^n}{n!} \]

\[ = \int_{M_t} \langle L_{v_i} L_{v_i} \Psi, \Psi \rangle \frac{\omega^n}{n!} + \int_{M_t} \langle L_{v_i} \Psi, L_{v_i} \Psi \rangle \frac{\omega^n}{n!}. \]

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III. Idea of proof of (a)

Generalizations: Fix \( v = v_i \).

Define \( \Psi_J \):= \( H(\Phi \cdots \Phi) \), \( ℓ \)-times. 

\[
\text{Harmonic part of } A.
\]

\[
\frac{∂}{∂i} \log Ψ_J^2 = 1 \|Ψ_J\|^2^2
\]

\[
\begin{align*}
&-k(\Box - k - 1) \langle Φ_i, Φ \rangle \langle Ψ_J, Ψ_J \rangle \\
&-k(\Box - k - 1) \langle L v_i Ψ_J, L v_i Ψ_J \rangle \\
&-\|L v_i Ψ_J\|_2^2
\end{align*}
\]

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\[ H(A) \text{ harmonic part of } A. \]

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\partial_i \partial_i \log \| \Psi_J \|_2 = 1 \| \Psi_J \|_2^2 \]

\[
- k ((\Box - k) - 1 (\Phi_i \cdot \Psi_J) - k ((\Box - k) - 1 \langle \Phi_i, \Phi_i \rangle, \langle \Psi_J, \Psi_J \rangle) - k ((\Box - k) - 1 (L_v i \Psi_J), L_v i \Psi_J) - \bigg\| (L_v i \Psi_J, \Psi_J) \bigg\|^2 - (H(\Phi_i \Psi_J), H(\Phi_i \Psi_J)).
\]
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  $H(A)$: harmonic part of $A$.

\[
\partial_i \overline{\partial_i} \log \| \Psi_j \|^2_2 = \frac{1}{\| \Psi_j \|^2_2} \left( - k((\Box - k)^{-1}(\Phi_i \cdot \Psi_j), \overline{\Phi_i} \cdot \Psi_j) \\
- k((\Box - k)^{-1}\langle \Phi_i, \Phi_i \rangle, \langle \Psi_j, \Psi_j \rangle) \\
- k((\Box - k)^{-1}(\mathcal{L}_{v_i} \Psi_j), \mathcal{L}_{v_i} \Psi_j) \\
- \left| (\mathcal{L}_{v_i} \Psi_j, \frac{\Psi_j}{\| \Psi_j \|^2_2}) \right|^2 \\
- (H(\Phi_i \otimes \Psi_j), H(\Phi_i \otimes \Psi_j)). \right)
\]
III. Idea of proof of (a)

Here $\Phi_i \cdot \Psi_J \in A_0^{\ell - 1} (\wedge \ell - 1)_{TM_t}$ has components given by

$$(\Phi_i \cdot \Psi_J)_{\alpha_1 \cdots \alpha_{\ell - 1} \beta_1 \cdots \beta_{\ell - 1}} = (\Phi_i)_{\sigma \gamma} \cdot (\Psi_J)_{\gamma \alpha_1 \cdots \alpha_{\ell - 1} \sigma \beta_1 \cdots \beta_{\ell - 1}}.$$

To obtain the identity:

Need various integration by parts
Regrouping of terms guided by geometry
Completing of squares (Bochner type arguments).

In retrospect, a similar expression was obtained independently by Schumacher (12) in a slightly different form.
Here $\Phi_i \cdot \Psi_J \in A^{0,\ell-1}(\wedge^{\ell-1}TM_t)$ has components given by

$$(\Phi_i \cdot \Psi_J)_{\alpha_1 \ldots \alpha_{\ell-1}}^{\beta_1 \ldots \beta_{\ell-1}} = (\Phi_i)^{\sigma}_{\gamma} \cdot (\Psi_J)^{\gamma \alpha_1 \ldots \alpha_{\ell-1}}_{\sigma \beta_1 \ldots \beta_{\ell-1}}.$$
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(\Phi_i \cdot \Psi_J)^{\alpha_1\cdots\alpha_{\ell-1}}_{\beta_1\cdots\beta_{\ell-1}} = (\Phi_i)^{\sigma}_{\gamma} \cdot (\Psi_J)^{\gamma\alpha_1\cdots\alpha_{\ell-1}}_{\bar{\sigma}\bar{\beta}_1\cdots\bar{\beta}_{\ell-1}}.
$$

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  \[
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- In retrospect, a similar expression was obtained independently by Schumacher (12) in a slightly different form.
III(ia). Idea of proof of (a)

Let $\Psi_J = \|\Psi_J\|_2^2$. Then

- Level 1: $\partial_v \log h(1) \geq h(1) - h(2) h(1)$
- Level 2: $\partial_v \log h(2) \geq h(2) - h(3) h(2)$
- Level n: $\partial_v \log h(n) \geq h(n) - h(n+1) h(n)$
III(ia). Idea of proof of (a)

- The above implies

\[ \partial_i \bar{\partial}_i \log \| \Psi_J \|_2^2 \geq \frac{\| \Psi_J \|_2^2}{\| H^{(\ell-1)} \|_2^2} - \frac{\| H^{(\ell+1)} \|_2^2}{\| \Psi_J \|_2^2}. \]
III(ia). Idea of proof of (a)

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\[
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\]

- Let \( \Psi_J = \| \Psi_J \|_2^2 \). Then

\[
\begin{align*}
\text{level 1} & \quad \partial_v \bar{\partial}_v \log h^{(1)} \geq \frac{h^{(1)}}{h^{(0)}} - \frac{h^{(2)}}{h^{(1)}} \\
\text{level 2} & \quad \partial_v \bar{\partial}_v \log h^{(2)} \geq \frac{h^{(2)}}{h^{(1)}} - \frac{h^{(3)}}{h^{(2)}} \\
\text{\ldots} & \quad \text{\ldots}
\end{align*}
\]

\[
\text{level n} \quad \partial_v \bar{\partial}_v \log h^{(n)} \geq \frac{h^{(n)}}{h^{(n-1)}} - \frac{h^{(n+1)}}{h^{(n)}}
\]
III. Idea of proof of (a)

But \( h(n+1) \in H(n+1) \wedge M(t) = 0 \).

Use good term on level \( i \) to control bad term on level \( i-1 \).

**Proposition**

Let \( \sigma = \max \{ \ell : \Psi_J \neq 0 \} \), \( N = n! \), \( C_1 = \min \{ 1, k_n n! (2\pi) nK_m \} \), \( C_\sigma = \sigma^{1/3} \sigma^{-1} a_\ell \), \( a_\ell = \left( 3C_1 \right)^N (N\ell^{-1} - 1)^N \).

Then for \( h(v, v) := \left( \sigma \sum_{\ell=1} a_\ell \|\Psi_J\|_2^{N/\ell} \right)^{1/2} \),

\[ \frac{\partial v}{\partial v} \log h(v, v) \geq C_\sigma^{1/N} a_1^{1/N} h(v, v). \]

**Remark**

For \( n = 1 \), get back the results for Riemann surfaces.

**Remark**

Note that the sum stops at \( \sigma \), which is important for Part III(ii).
III. Idea of proof of (a)

- But $h^{(n+1)} \in H^{n+1}(\wedge^{n+1} T_{M_t}) = 0$. 

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Let $\sigma = \max\{\ell : \Psi_J \neq 0\}$, $N = n!$, $C_1 = \min\{1, \frac{k^n n!}{(2\pi)^n K_{M_t}^n}\}$,

$$C_\sigma = \frac{\sigma^1}{3\sigma - 1}, \ a_\ell = \left( \frac{3}{C_1} \right)^{\frac{N(N^{\ell-1} - 1)}{N - 1}}.$$ Then for

$$h(v, \bar{v}) := \left( \sum_{\ell=1}^{\sigma} a_\ell \|\Psi_J\|_2^{2N/\ell} \right)^{1/2N},$$

$$\partial_v \bar{\partial}_v \log h(v, \bar{v}) \geq \frac{C_\sigma}{\sigma^{1/N} a_\sigma^{1 + 1/N}} h(v, \bar{v}).$$
III. Idea of proof of (a)

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Let $\sigma = \max\{\ell : \Psi_J \neq 0\}$, $N = n!$, $C_1 = \min\{1, \frac{k^n n!}{(2\pi)^n K_{M_t}^n}\}$,

$$C_\sigma = \frac{\sigma_1}{3^{\sigma - 1}}, a_\ell = \left(\frac{3}{C_1}\right)^{\frac{N(N\ell - 1 - 1)}{N - 1}}.$$ Then for

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III. Idea of proof of (a)

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  Let $\sigma = \max\{\ell : \Psi J \neq 0\}$, $N = n!$, $C_1 = \min\{1, \frac{k^n n!}{(2\pi)^n K_M^n}\}$,
  
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  $$h(v, \bar{v}) := \left(\sum_{\ell=1}^{\sigma} a_\ell \|\Psi J\|_2^{2N/\ell}\right)^{1/2N},$$
  
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- **Remark** For $n = 1$, get back the results for Riemann surfaces.

- **Remark** Note that the sum stops at $\sigma$, which is important for Part III(ii).
III(ib). Idea of proof of (b)

Consider a family $\pi: X \to S$ with fiber $(M_t, \omega_t)$, where $M_t$ is Kähler Ricci flat, $\omega_t$ polarization. Require: cohomology class $[\phi^* \omega_t] \in H^2(M_0, \mathbb{C})$ is constant.

Here $\phi_t: M_0 \to M_t$ is induced from a smooth trivialization $\phi: M_0 \times I \to X$.

Analogous to the work of Siu, Nannicini (86) obtained:

$$R(WP)_{\bar{i} \bar{j} \bar{k} \bar{l}}(t) = -\frac{1}{4} V(h_{ij} h_{kl} + h_{ik} h_{jl}) (1)$$

$$- \int_{M_t} \langle L v_i \Phi_k, L v_j \Phi_\ell \rangle \omega^n + \int_{M_t} \langle H(\Phi_i \Phi_k), H(\Phi_j \Phi_\ell) \rangle \omega^n,$$

here $V$ is the volume of $M_0$. 

III(ib). Idea of proof of (b)

Consider a family $\pi : \mathcal{X} \to S$ with fiber $(M_t, \omega_t)$, where $M_t$ is Kähler Ricci flat, $\omega_t$ polarization.
III(ib). Idea of proof of (b)

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III(ib). Idea of proof of (b)

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$$R_{i\bar{j}k\ell}^{(WP)}(t) = -\frac{1}{4V} (h_{i\bar{j}}h_{l\bar{k}} + h_{i\bar{k}}h_{l\bar{j}})$$

$$- \int_{M_t} \langle (\mathcal{L}_{v_i} \Phi_k, \mathcal{L}_{v_j} \Phi_\ell) \rangle \frac{\omega^n}{n!}$$

$$+ \int_{M_t} \langle H(\Phi_i \otimes \Phi_k), H(\Phi_j \otimes \Phi_\ell) \rangle \frac{\omega^n}{n!},$$

here $V$ is the volume of $M_o$. 


IV. Idea of proof of (b)

To handle the last term, for $\Psi_J := H(\Phi_j^1 \cdots \Phi_j^\ell) \in A_0^\ell \land TM_t$, we prove

$$\partial_i \partial_i \log \|\Psi_J\|^2_2 = 1 \|\Psi_J\|^2_2$$

$$\partial_i \partial_i \log \|\Psi_J\|^2_2 = \left( H(\Phi_i \cdot \Psi_J), \Phi_i \cdot \Psi_J \right) + \left( H(\langle \Phi_i, \Phi_i \rangle), \langle \Psi_J, \Psi_J \rangle \right) + \left( H(Lv_i \Psi_J), Lv_i \Psi_J \right) - \left| \left( Lv_i \Psi_J, \Psi_J \right) \right|^2_2 - \left( H(\Phi_i^? \Psi_J), H(\Phi_i^? \Psi_J) \right).$$

Use bootstraping argument to construct a Finsler metric of negative holomorphic sectional curvature.
IV. Idea of proof of (b)

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$$\partial_i \bar{\partial}_i \log \|\Psi_J\|^2_2 = \frac{1}{\|\Psi_J\|^2_2} \left( H(\bar{\Phi}_i \cdot \Psi_J, \bar{\Phi}_i \cdot \Psi_J) + (H(\langle \Phi_i, \Phi_i \rangle, \langle \Psi_J, \Psi_J \rangle) + ((H(\mathcal{L}_{v_i} \Psi_J, \mathcal{L}_{v_i} \Psi_J) - \left| (\mathcal{L}_{v_i} \Psi_J, \frac{\Psi_J}{\|\Psi_J\|_2}) \right|^2 - (H(\Phi_i \ominus \Psi_J), H(\Phi_i \ominus \Psi_J))).$$
IV. Idea of proof of (b)

To handle the last term, for

\[ \Psi_J := H(\Phi_{j_1} \otimes \cdots \otimes \Phi_{j_\ell}) \in A^{0,\ell}(\wedge^\ell TM_t) \]

we prove

\[
\partial_i \overline{\partial_i} \log \|\Psi_J\|_2^2 = \frac{1}{\|\Psi_J\|_2^2} \left( H(\overline{\Phi_i} \cdot \Psi_J, \overline{\Phi_i} \cdot \Psi_J) + (H(\langle \Phi_i, \Phi_i \rangle), \langle \Psi_J, \Psi_J \rangle) \right) \\
+ \left( (H(\mathcal{L}_v \Psi_J, \mathcal{L}_v \Psi_J) - \|\mathcal{L}_v \Psi_J, \frac{\Psi_J}{\|\Psi_J\|_2}\|_2^2 \right) \\
- (H(\Phi_i \otimes \Psi_J, H(\Phi_i \otimes \Psi_J))).
\]

Use bootstrapping argument to construct a Finsler metric of negative holomorphic sectional curvature.
IV. About the proof of (b)

Remark
Candelas, de la Ossa, Green and Parkes constructed a family of Calabi-Yau threefolds with mixed signs in the curvature of $g_{WP}$. Hence higher order augmented metric cannot be avoided.

The same scheme works for orbifolds. Need to make sure that Hodge Decomposition, Green's kernels make sense for orbifolds.
IV. About the proof of (b)

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- **Remark** Candelas, de la Ossa, Green and Parkes constructed a family of Calabi-Yau threefolds with mixed signs in the curvature of $g_{WP}$. Hence higher order augmented metric cannot be avoided.

- The same scheme works for orbifolds. Need to make sure that Hodge Decomposition, Green’s kernels make sense for orbifolds.
III(ic). About the proof of (c)

Technical difficulties:

1. Non-compact fibers, need to make sure that integration by parts make sense.
2. Need to make sure that Hodge Decomposition, Spectral Decomposition make sense for the special class of non-compact manifolds that we study (log-canonically polarized).
3. Need to use some sort of Maximum Principle for complete non-compact manifolds.
4. The above for tensors obtained after Lie derivatives with respect to the canonical (horizontal) lifts.
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III(ii). About the proof of (generalized) Viehweg Conjecture
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- Proposition

There exists a Viehweg-Zuo sheaf in cases (a), (b), (c)
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» Idea of Proof
III(ii). About the proof of (generalized) Viehweg Conjecture

- **Proposition**

  There exists a Viehweg-Zuo sheaf in cases (a), (b), (c)

- **Idea of Proof**

  - Consider first a Zariski open set $U$ of $\mathcal{M}$ on which it is effectively parametrized.
III(ii). About the proof of (generalized) Viehweg Conjecture

- **Proposition**

  There exists a Viehweg-Zuo sheaf in cases (a), (b), (c)

- **Idea of Proof**

  - Consider first a Zariski open set $U$ of $M$ on which it is effectively parametrized.
  - Take a basis $\frac{\partial}{\partial t^1}, \cdots, \frac{\partial}{\partial t^m}$ of $T_t S$, and let $\Phi_i$ be the harmonic representative of $\rho_t(\frac{\partial}{\partial t^i})$ on $M_t$ as before.
III(ii). About the proof of (generalized) Viehweg Conjecture

▶ Proposition

There exists a Viehweg-Zuo sheaf in cases (a), (b), (c)

▶ Idea of Proof

▶ Consider first a Zariski open set $U$ of $\mathcal{M}$ on which it is effectively parametrized.

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▶ Consider the map $\rho^{(\ell)}_t : S^\ell(T_t S) \to \mathcal{A}^{0,\ell}(\wedge^\ell TM_t)$ given by

$$\rho^{(\ell)}_t \left( \frac{\partial}{\partial t^{j_1}} \otimes \cdots \otimes \frac{\partial}{\partial t^{j_\ell}} \right) = \Psi_J := H(\Phi_{j_1} \otimes \cdots \otimes \Phi_{j_\ell}).$$
Let $1 < \sigma \leq n$ be the smallest integer $\ell$ such that $\rho^{(\ell+1)} = 0$ identically on $S$. 
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Consider

$$0 \to \ker \rho^{(\ell)} \to S^\ell(T_S) \to S^\ell(T_S)/\ker \rho^{(\ell)} \to 0. \quad (2)$$
Let $1 < \sigma \leq n$ be the smallest integer $\ell$ such that $\rho^{(\ell+1)} = 0$ identically on $S$.

Consider

$$0 \to \ker \rho^{(\ell)} \to S^\ell(T_S) \to S^\ell(T_S)/\ker \rho^{(\ell)} \to 0. \tag{2}$$

$\mathcal{V} = (S^\sigma(T_S)/\ker \rho^{(\sigma)})^*$ is a coherent subsheaf of $S^\ell(\Omega_S)$. 
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$\mathcal{V} = (S^\sigma(T_S)/\ker \rho^{(\sigma)})^*$ is a coherent subsheaf of $S^\ell(\Omega_S)$.

$\mathcal{V}$ is a vector bundle on a Zariski open set $U_\sigma$ of $S$. 
Let $1 < \sigma \leq n$ be the smallest integer $\ell$ such that $\rho^{(\ell+1)} = 0$ identically on $S$.

Consider

$$0 \to \ker \rho^{(\ell)} \to S^\ell(T_S) \to S^\ell(T_S)/\ker \rho^{(\ell)} \to 0. \quad (2)$$

$V = (S^\sigma(T_S)/\ker \rho^{(\sigma)})^*$ is a coherent subsheaf of $S^\ell(\Omega_S)$.

$\mathcal{V}$ is a vector bundle on a Zariski open set $U_\sigma$ of $S$.

$g_{WP,\sigma}$ is non-degenerate on $\mathcal{V}$ from definition.
Computation shown earlier for $\Psi_J$ on $U_\ell$ gives,

\[
\frac{\partial}{\partial i} \log \| \Psi_J \|^2 = 1 \| \Psi_J \|^2
\]

\[
- k \left( (\Box - k) - 1 (\Phi_i \cdot \Psi_J) \right) - k \left( (\Box - k) - 1 \langle \Phi_i, \Phi_i \rangle, \langle \Psi_J, \Psi_J \rangle \right)
\]

\[
- k \left( (\Box - k) - 1 (Lv_i \Psi_J), Lv_i \Psi_J \right) - \left| \left| (Lv_i \Psi_J), \Psi_J \right| \right|^2
\]

For $\ell = \sigma$, the last term $H(\Phi_i \Psi_J) = 0$. 
Computation shown earlier for $\Psi_J$ on $U_\ell$ gives,

\[
\partial_i \overline{\partial_i} \log \| \Psi_J \|^2_2 = \frac{1}{\| \Psi_J \|^2_2} \left( - k ((\Box - k)^{-1}(\Phi_i \cdot \Psi_J), \overline{\Phi_i} \cdot \Psi_J) 
- k ((\Box - k)^{-1}\langle \Phi_i, \Phi_i \rangle, \langle \Psi_J, \Psi_J \rangle) 
- k ((\Box - k)^{-1}(L_{\nu_i} \Psi_J), L_{\nu_i} \Psi_J) 
- \left| (L_{\nu_i} \Psi_J, \frac{\Psi_J}{\| \Psi_J \|^2_2}) \right|^2 
- (H(\Phi_i \odot \Psi_J), H(\Phi_i \odot \Psi_J)) \right).
\]
Computation shown earlier for $\Psi_J$ on $U_\ell$ gives,

\[
\partial_i \overline{\partial_i} \log \|\Psi_J\|^2 \quad = \quad \frac{1}{\|\Psi_J\|^2} \left( - k \langle (\Box - k)^{-1} (\Phi_i \cdot \Psi_J), \Phi_i \cdot \Psi_J \rangle \\
- k \langle (\Box - k)^{-1} \langle \Phi_i, \Phi_i \rangle, \langle \Psi_J, \Psi_J \rangle \rangle \\
- k \langle (\Box - k)^{-1} (\mathcal{L}_{\nu_i} \Psi_J), \mathcal{L}_{\nu_i} \Psi_J \rangle \\
- \left( \mathcal{L}_{\nu_i} \Psi_J, \frac{\Psi_J}{\|\Psi_J\|_2} \right)^2 \\
- (H(\Phi_i \cdot \Psi_J), H(\Phi_i \cdot \Psi_J)) \right).
\]

For $\ell = \sigma$, the last term $H(\Phi_i \cdot \Psi_J) = 0$. 
It follows that
\[ \partial_i \partial_i \log \| \Psi_\ell \|_2^2 \geq \| \Psi_\ell \|_2^2 \left( -k \left( \Box - k \right) - 1 \langle \Phi_i, \Phi_i \rangle, \langle \Psi_\ell, \Psi_\ell \rangle \right) \] (3)

Hence
\[ \partial_i \partial_i \log \| \Psi_\ell \|_2^2 \geq \| \Psi_\ell \|_2^2 \left( \int_{x \in M} \langle \nu_i, \nu_i \rangle \| \Psi_\ell \|_2^2 (x) \right) > 0 \] (4)
where \( \nu_i \) is the canonical lift of \( \Phi_i \).

We get a Griffith positive subsheaf \( V \) of \( S_\ell(\Omega S) \).
It follows that

\[ \partial_i \bar{\partial}_i \log \| \psi_\ell \|^2 \geq \frac{1}{\| \psi_\ell \|^2} \left( - k ((\Box - k)^{-1} \langle \Phi_i, \Phi_i \rangle, \langle \psi_\ell, \psi_\ell \rangle) \right). \]

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\partial_i \bar{\partial}_i \log \| \psi_\ell \|_2^2 \geq \frac{1}{\| \psi_\ell \|_2^2} \left( \int_{x \in M_t} \langle \nu_i, \nu_i \rangle \| \psi_\ell \|^2(x) \right) > 0 
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\[
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where \( \nu_i \) is the canonical lift of \( \Phi_i \).

We get a Griffith positive subsheaf \( \mathcal{V} \) of \( S^\ell(\Omega_S) \).
Standard $L^2$-estimates allow us to construct a lot of sections for $V$ on $U_\ell$, hence bigness on $U_\ell$.

As explained, $g_{WP,\ell}$ is non-degenerate on $V$. Riemann Extension Theorem allows us to extend $L^2$ sections from $U_\ell$ to $S$.

To extend the sheaf $V$ across $S - S$ is more difficult.

For this we used Theorem 1a, -ve hol sectional curv, to estimate the augmented Finsler metric by the Poincaré metric $g_P$ in a neighborhood of $D$, using Ahlfors Schwartz Lemma.

This in terms bounds Weil-Petersson metric $g_{WP,1}$ by $g_P$, from which we can show that $L^2$ sections of $V|_S$ extends as log sections to $S$ to conclude Proposition 1.
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To extend the sheaf $\mathcal{V}$ across $S$ is more difficult.

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Standard $L^2$-estimates allow us to construct a lot of sections for $\mathcal{V}$ on $U_\ell$, hence bigness on $U_\ell$.

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To extend the sheaf $\mathcal{V}$ across $\overline{S} - S$ is more difficult.
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- Standard $L^2$-estimates allow us to construct a lot of sections for $\mathcal{V}$ on $U_\ell$, hence bigness on $U_\ell$.
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- To extend the sheaf $\mathcal{V}$ across $\overline{S} - S$ is more difficult.
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- This in terms bounds Weil-Petersson metric $g_{WP,1}$ by $g_P$, from which we can show that $L^2$ sections of $\mathcal{V}|_S$ extends as log sections to $\overline{S}$ to conclude Proposition 1.
Idea for Proof of Theorem 2.

Once we have Proposition, we can use the results of Campana-Paun or modify Miyaoka’s generic semi-negativity Theorem to conclude that $K_X + D$ is big. Hence Theorem 2 for Case (a).

Appropriate modifications of the arguments can be applied to (b) and (c).
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Idea for Proof of Theorem 2.

Once we have Proposition, we can use the results of Campana-Paun or modify Miyaoka’s generic semi-negativity Theorem to conclude that $K_S + D$ is big. Hence Theorem 2 for Case (a).

Appropriate modifications of the arguments can be applied to (b) and (c).