Extremal Particles in Branching Brownian Motion - Part 1

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Overview:

Part 1: Extremal Particles in Branching Brownian Motion.
Part 2: Branching Brownian Motion under Selection.

Part 1

- Branching Brownian motion (BBM) - Definition and basic properties.
- The F-KPP equation and its connection to BBM.
- The distribution of the maximum of BBM.
- Extremal particles in BBM
  - Genealogy of extremal particles of BBM (cf. Arguin, Bovier and Kistler, [ABK11]);
  - Poissonian statistics in the extremal process of BBM (cf. [ABK12]).

Remark:

Following the bibliography, further slides are given relating to

- The extremal process of BBM (cf. [ABK13]);
- BBM seen from its tip (cf. Aïdékon, Berestycki, Brunet and Shi, [ABBS13]);
- Additional details, with pointers given by ⇓.
Branching Brownian Motion

Definition (Branching Brownian Motion (BBM))

- \( t = 0 \): single particle \( x_1(0) \) starts at origin;
- moves as a Brownian motion (BM) in \( \mathbb{R}^1 \) until
- after (at) time \( \tau \sim \text{Exp}(1) \) it splits into
- two identical particles that start (both) at \( x_1(\tau) \) and
- move as two independent BMs each.
- Repeat.

The resulting process is a collection of a (random) number \( n(t) \) of particles

\[
(x_1(t), x_2(t), \ldots, x_{n(t)}(t))_{t\geq 0} = \{x_k(t) : k \leq n(t)\}_{t\geq 0}.
\]

Note 1. \( n(\tau) = 2 \).
Note 2. Here we use w.l.o.g. binary branching.
(see homepage of Matt Roberts, Univ. of Bath)
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We have seen $M(t) = \max_{u \in N_t} X_u(t)$, the position of the rightmost particle can be studied through the analysis of the KPP equation. In this section we start our exploration of the extremal point process of the branching Brownian motion by looking at the asymptotic behavior of $M(t)$.

As we have seen in the previous chapter, $u(t, x) = P_0(M(t) \leq x)$ solves the F-KPP equation with initial condition $u(0, x) = 1\{x > 0\}$. The following is one of the results proved in the original paper of Kolmogorov et al.

(see J. Berestycki (Lecture Notes, "Topics on BBM", http://www.stats.ox.ac.uk/~berestyc/Articles/EBP18_v2.pdf), Image by Matt Roberts)
Remark (see Bovier, [B15] for more respectively for references to literature)

1. There are connections to spin glass theory; in particular, Generalised Random Energy models (GREM).

2. Many results can be extended to branching random walk.

3. Connection to extremes of the free Gaussian random field in $d = 2$.

4. Can be extended to variable speed BBM.

Remark (Genealogies of BBM)

Let

$$d(x_k(t), x_\ell(t)) \equiv \inf\{0 \leq s \leq t : x_k(s) \neq x_\ell(s)\} = \text{time (from 0) to MRCA}$$

$$\equiv \text{unique time where the most recent common ancestor split}$$

$$= \text{time of death of longest surviving ancestor of both particles.}$$

A BBM can then also be constructed as follows. Construct first a continuous time Galton-Watson tree with binary branching. Let $I(t)$ be the set of its leaves at time $t$. Then, conditional on the GW-process, BBM is a Gaussian process $x_k(t), k \in I(t)$ with

$$\mathbb{E}[x_k(t)] = 0 \text{ and } \text{Cov}(x_k(t), x_\ell(t)) = d(x_k(t), x_\ell(t)).$$
First Properties

1. \( \mathbb{E}[n(t)] = e^t. \)

2. \( e^{-t}n(t) \) is a martingale that converges, a.s. and in \( L^1 \), to an exponential r.v. of parameter 1.
The F-KPP equation

Consider the partial differential equation

\[ u_t = \frac{1}{2} u_{xx} + u^2 - u, \quad u = u(t, x) \in [0, 1], \quad t \geq 0, \quad x \in \mathbb{R}, \quad u(0, x) = f(x). \quad (1) \]

Set \( v = 1 - u \). Then this is a special case of the Kolmogorov-Petrovskii-Piskunov-(KPP)-equation (also known as the Kolmogorov- or Fisher-equation).

Let \( \{x_k(t) : k \leq n(t)\}_{t \geq 0} \) be a BBM starting at 0 and \( f : \mathbb{R} \to [0, 1] \). Then

\[ u(t, x) = \mathbb{E} \left[ \prod_{k=1}^{n(t)} f(x - x_k(t)) \right] \]

is the solution to the F-KPP equation (1) with \( u(0, x) = f(x) \).
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Idea of proof: Branching property: Let \( p_t(x) = \frac{1}{\sqrt{2\pi t}} e^{-x^2/2t} \) denote the Heat kernel. Then (use that \( \tau \sim \text{Exp}(1) \), i.e. \( f_\tau(s) = 1_{\{s \geq 0\}} e^{-s} \) and \( \mathbb{P}(\tau > t) = e^{-t} \))

\[ u(t, x) = e^{-t} \int p_t(z)f(x - z)dz + \int_0^t e^{-s} \int p_s(z)u^2(t - s, x - z)dzds. \]

Now differentiate w.r.t. \( t \), use integration by parts and \( \frac{\partial}{\partial t} p_t(x) = \frac{\partial^2}{\partial x \partial x} \left( \frac{p_t(x)}{2} \right) \). \( \square \)
\[ u_t = \frac{1}{2} u_{xx} + u^2 - u, \quad u(0, x) = f(x) \] has as solution 

\[ u(t, x) = \mathbb{E} \left[ \prod_{k=1}^{n(t)} f(x - x_k(t)) \right] \]

**Example 1.** Let \( M(t) \equiv \max_{k \leq n(t)} x_k(t) \). With \( f(x) = 1_{[0, \infty)}(x) \) we obtain

\[
u(t, x) = \mathbb{E} \left[ \prod_{k=1}^{n(t)} 1_{[0, \infty)}(x - x_k(t)) \right] = \mathbb{E} \left[ \prod_{k=1}^{n(t)} 1_{\{x_k(t) \leq x\}} \right] = \mathbb{P} \left( \max_{k \leq n(t)} x_k(t) \leq x \right) = \mathbb{P}(M(t) \leq x) = F_{M(t)}(x).

**Example 2.** Let \( f(x) = e^{-\phi(x)}, \phi \in C_c^+ \). Set \( P_t \equiv \sum_{k=1}^{n(t)} \delta_{x_k(t)} \). Then

\[
u(t, x) = \mathbb{E} \left[ \prod_{k=1}^{n(t)} e^{-\phi(x - x_k(t))} \right] = \mathbb{E} \left[ e^{-\int \phi(x - z) P_t(dz)} \right].\]
The F-KPP equation revisited

Consider $v(t, x) \equiv 1 - u(t, x)$ instead of $u(t, x)$. Then $v(t, x)$ solves

$$v_t = \frac{1}{2} v_{xx} - v^2 + v = \frac{1}{2} v_{xx} + v(1 - v), \quad v(0, x) = 1 - u(0, x). \quad (2)$$

The Feynman-Kac formula yields the following representation for the linear equation

$$v_t = \frac{1}{2} v_{xx} + k(t, x) v, \quad v_0(x) = v(0, x).$$

Namely,

$$v(t, x) = \mathbb{E} \left[ \exp \left( \int_0^t k(t - s, B^x(s)) ds \right) v(0, B^x(t)) \right]$$

$$= \mathbb{E}_x \left[ \exp \left( \int_0^t k(t - s, B(s)) ds \right) v(0, B(t)) \right].$$

Here, $(B^x(t))_{t \geq 0}$ is a BM, starting in $x \in \mathbb{R}$.

Bramson [B83] sets $k(t, x) \equiv 1 - v(t, x)$ with $v$ solving (2). Then

$$v(t, x) = \mathbb{E}_x \left[ \exp \left( \int_0^t (1 - v(t - s, B(s))) ds \right) v(0, B(t)) \right]$$

(“implicit description of $v$”).
\[ v(t, x) = \mathbb{E}_x \left[ \exp \left( \int_0^t k(t-s, B(s)) \, ds \right) v(0, B(t)) \right] \]

\[ = \mathbb{E}_x \left[ \exp \left( \int_0^t (1 - v(t-s, B(s))) \, ds \right) v(0, B(t)) \right]. \]

**Observations:** (recall: \( 1 - v(t, x) = u(t, x), \ u(0, x) = 1_{[0, \infty)}(x) \))

- "\( v(t, x) \) is the weighted average of the different sample paths of BM."
- \( 0 < k(t - s, B(s)) < 1, \)
- can show: \( \lim_{y \to -\infty} k(r, y) \leq \epsilon \) for \( r \geq r(\epsilon), \ \lim_{y \to \infty} k(r, y) = 1; \)
- i.e., for \( y \) large, weighting of path nearly maximal, for \( y \) small, insignificant.
- Transition near \( k(r, y) = 1/2 \iff u(r, y) = 1/2 \)
  (preview: \( m(t) = \sup \{ x : u(t, x) \leq 1/2 \} + O(1) \)).

Bramson [B83] distinguishes paths \( x(s), 0 \leq s \leq t \) according to

\[ \exp \left( \int_0^t k(t-s, x(s)) \, ds \right) \sim e^t \quad \text{or} \quad \exp \left( \int_0^t k(t-s, x(s)) \, ds \right) \ll e^t. \]
How to calculate $v(t, x)$?

$$v(t, x) = \mathbb{E} \left[ \exp \left( \int_0^t k(t - s, B^x(s)) ds \right) v(0, B^x(t)) \right]$$

$$= \mathbb{E} \left[ \mathbb{E} \left[ \exp \left( \int_0^t k(t - s, B^x(s)) ds \right) v(0, B^x(t)) \mid B^x(t) \right] \right]$$

$$= \int_{-\infty}^{\infty} v(0, y) \frac{e^{-(x-y)^2/2t}}{\sqrt{2\pi t}} \mathbb{E} \left[ \exp \left( \int_0^t k(t - s, \delta^{t}_{x,y}(s)) ds \right) \right] dy,$$

where $\delta^{t}_{x,y}$ denotes a Brownian bridge starting at $x$ (at time 0) and ending at $y$ (at time $t$).

- A Brownian bridge has the distribution of a BM starting at $x$, conditional on being in $y$ at time $t$.
- $\delta^{t}_{0,0}(s) \equiv B^0(s) - \frac{s}{t} B^0(t)$, $0 \leq s \leq t$ is
  - a Gaussian process (all finite-dim. distrib.s are normally distributed),
  - a.s. continuous on $[0, t]$, a strong Markov process and indep. of $B^0(t)$.
- $\delta^{t}_{x,y}(s) \overset{D}{=} \delta^{t}_{0,0}(s) + \frac{s}{t} y + \frac{t-s}{t} x$, $0 \leq s \leq t$.
- $\text{Var}(\delta^{t}_{0,0}(s)) = \frac{s(t-s)}{t}$ with a maximum in the middle, $\text{Var}(\delta^{t}_{0,0}(t/2)) = t/4$. 
(Brownian Bridge, cf. https://www.researchgate.net/figure/228766780_fig2_

Figure-Sample-path-examples-of-a-Brownian-bridge-for-different-initial-and-final-states)
The distribution of the maximum of BBM

Let us return to Example 1. Then \( u(t, x) = \mathbb{P}(M(t) \leq x) = \mathbb{P}\left( \max_{k \leq n(t)} x_k(t) \leq x \right) \) solves (1) with \( u(0, x) = 1_{[0, \infty)}(x) \).

As a result: \( 0 < u(t, x) < 1 \) for all \( x \in \mathbb{R}, t > 0 \).

Take as centering term,

\[
m(t) \equiv \sqrt{2}t - \frac{3}{2\sqrt{2}} \log(t), \quad \text{(in i.i.d. case} \quad \sqrt{2}t - \frac{1}{2\sqrt{2}} \log(t) \overset{\mathcal{D}}{\to} 1)\]

then \( m(t) = \sup\{x : u(t, x) \leq 1/2\} + O(1) \) (cf. Bramson [B83], Roberts [R13]) and

\[
\mathbb{P}(M(t) - m(t) \leq x) = u(t, m(t) + x) \to w(x) \quad \text{unif. in } x \text{ as } t \to \infty. \quad (3)
\]

Here, \( w(x) \) is the unique (up to translation) solution of the equation

\[
\frac{1}{2} w_{xx} + \sqrt{2} w_x + w^2 - w = 0
\]

satisfying \( 0 < w(x) < 1 \) for all \( x \in \mathbb{R} \) and \( w(x) \to 0 \) as \( x \to -\infty \), \( w(x) \to 1 \) as \( x \to \infty \).
The derivative martingale

Lalley and Sellke [LS87] show: Let

\[ Z(t) \equiv \sum_{k=1}^{n(t)} \left( \sqrt{2}t - x_k(t) \right) e^{-\sqrt{2} \left( \sqrt{2}t - x_k(t) \right)} \]

be the so-called derivative martingale, then

\[ Z = \lim_{t \to \infty} Z(t) \]  \hspace{1cm} (4)

exists and is strictly positive a.s. Moreover, for some \( C > 0, \)

\[
P(M(t) - m(t) \leq x) \xrightarrow{t \to \infty} \mathbb{E} \left[ \exp \left( -CZ e^{-\sqrt{2}x} \right) \right] = \mathbb{E} \left[ \exp \left( -e^{-\sqrt{2} \left( x - \frac{\log(CZ)}{\sqrt{2}} \right)} \right) \right].\]  \hspace{1cm} (5)

Note 1. The so-called Gumbel distribution has cumulative distribution function

\[ F_G(x) = \mathbb{P}(G \leq x) = \exp \left( -e^{-\left( x - \mu \right) / \beta} \right) = \exp \left( -e^{\mu / \beta} e^{-x / \beta} \right) \]

with parameters \( \mu \in \mathbb{R}, \beta > 0. \) Hence, \( w(x) \) represents a random shift of "the" Gumbel distribution with \( \mu = 0 \) and \( \beta = 1 / \sqrt{2}. \)

Note 2. \( 1 - w(x) \sim Cxe^{-\sqrt{2}x} \) for \( x \to \infty. \)
Additionally, [LS87] showed

**Theorem**

Suppose two independent BBMs \((X^A_1(t), \ldots, X^A_{n}(t))\) and \((X^B_1(t), \ldots, X^B_{n}(t))\) are started at 0 respectively \(x < 0\). Then, with probability 1, there exist finite random times \(t_n, n \in \mathbb{N}, t_n \to \infty\) such that

\[
M^A(t_n) < M^B(t_n)
\]

for all \(n \in \mathbb{N}\).

**Idea of proof.** Use (3), i.e. \(\lim_{t \to \infty} \mathbb{P}(M(t) - m(t) \leq x) = w(x)\). \(\square\)

**Corollary**

Every particle born in a BBM has a descendant particle in the "lead" at some future time.
Extremal particles in BBM (cf. [ABK11] and [ABK12])

- **Arguin, L.-P. and Bovier, A. and Kistler, N.**

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We are interested in the limit \((t \to \infty)\) of the extremal process

\[
\mathcal{E}_t \equiv \sum_{k \leq n(t)} \delta_{x_k(t) - m(t)} \equiv \sum_{k \leq n(t)} \delta_{x_k(t)}.
\]

**Recall:** The centering term \(m(t)\) satisfies

\[
m(t) = \sqrt{2}t - \frac{3}{2\sqrt{2}} \log(t) = \sup\{x : u(t, x) \leq 1/2\} + O(1).
\]

By the previous Remark on Genealogies of BBM, for a given realization of the branching, the genealogical distances

\[
d(x_k(t), x_\ell(t)) = \inf\{0 \leq s \leq t : x_k(s) \neq x_\ell(s)\} = \text{time (from 0) to MRCA}.
\]
Genealogy of extremal particles of BBM

Theorem (Genealogy of Extremal Particles, Theorem 2.1, [ABK11])

For any compact set $D \subset \mathbb{R}$,

$$\lim_{r \to \infty} \sup_{t > 3r} \mathbb{P}(\exists 1 \leq k, \ell \leq n(t) : \bar{x}_k(t), \bar{x}_\ell(t) \in D \text{ and } d(x_k(t), x_\ell(t)) \in (r, t - r)) = 0.$$ 

Conclusion: The MRCA of extremal particles at time $t$ splits/branches off with high probability at a time

1. in the interval $(0, r)$ (”very early branching”) or
2. in the interval $(t - r, t)$ (”very late branching”).
Brunet and Derrida also conjecture that the extremal process of BBM retains properties of Poisson point processes with exponential density, namely the invariance under superposition: the collection of a finite number of i.i.d. copies of the process, with possibly relative shifts, has the same law for the gaps as the process itself. Although we cannot prove any of the Brunet-Derrida conjectures, our results let them appear rather natural. In fact, Theorem 2.1 suggests the following picture for the extremal process of BBM, which is depicted in Figure 2.4.

First, the result does not rule out ancestry in the interval $[0, r]$ (in the limit of large times and for large enough $r$): this “free evolution” seems to naturally generate the derivative martingale appearing in the work of Lalley and Sellke’s [20]. Second, with ancestry over the period $[t - r, t]$ also being allowed, it is obvious that small grapes of length at most $r^{1/2}$, i.e., clusters of particles with very recent common ancestors, appear at the end of the time interval. This suggests that particles at the edge of BBM should be more densely packed than in the REM case, in agreement with (2.15). Finally, since the ancestors of the extremal particles evolved independently for most of the time (in the interval $[r, t]$), the extremal process must exhibit a structure similar to the Poisson process with exponential density: this makes plausible the invariance under superposition of the law of the gaps proposed by Brunet and Derrida.

In other words, the limiting extremal process of BBM seems to be given by a certain randomly shifted cluster point process. A rigorous analysis is however technically quite demanding, as it must take into account the self-similarity of BBM.
Poissonian statistics in the extremal process of BBM

W.l.o.g., order the (centered) particles in decreasing order, i.e.

\[ x_1(t) \geq x_2(t) \geq \cdots \geq x_{n(t)}(t). \]  \hspace{1cm} (6)

Let

\[ \overline{D}(t) = \{ \overline{D}_{k\ell}(t) \}_{k, \ell \leq n(t)} \equiv \left\{ \frac{d(x_k(t), x_\ell(t))}{t} \right\}_{k, \ell \leq n(t)}. \]

Definition

Let \( 0 < q < 1 \). The \( q \)-thinning \( \mathcal{E}_{t}^{(q)} \) of the pair \( (\mathcal{E}_t, \overline{D}(t)) \) is defined as follows:

- Consider the equivalence classes of particles alive at time \( t \) with MRCA at a time later than \( q \cdot t \), i.e. \( k \sim_q \ell \iff \overline{D}_{k\ell}(t) \geq q \).

- Select the maximal (according to (6)) particle within each class.

- Then \( \mathcal{E}_{t}^{(q)} \) is the point process of these representatives.

Note 1. This extends to \( q = q(t) \in (0, 1) \).

Note 2. The thinning map \( (\mathcal{E}_t, \overline{D}(t)) \mapsto \mathcal{E}_{t}^{(q)}(t) \) is continuous (on the space of pairs \((X, Q)\), \( X \) ordered positions, \( Q \) symm. matrix with entries in \([0, 1]\) and transitive op. \( Q_{ij} \geq q \)).
Theorem (Theorem 2, [ABK12])

For any $0 < q < 1$, the processes $\mathcal{E}_t^{(q)}$ converge in law to the same limit, $\mathcal{E}^0$. Also,

$$\lim_{r \to \infty} \lim_{t \to \infty} \mathcal{E}_t^{(1-r/t)} = \mathcal{E}^0.$$ 

Moreover, conditionally on $Z$, (the limit of the derivative martingale, cf. (4)),

$$\mathcal{E}^0 = \text{PPP}(CZ \sqrt{2} e^{-\sqrt{2}x} dx)$$

(PPP stands for ”Poisson Point Process”)

where $C > 0$ is the constant appearing in (5).

(Figure 2 in [ABK12])
Conclusion:
The particles at the frontier of BBM for large times can be constructed as follows:

1. set down so-called cluster extrema according to $E^0$, that is, according to a randomly shifted PPP with "exponential" ($x \in \mathbb{R}$) density;
2. attach to each cluster extrema a cluster.

Note 1. Particles in one cluster lie to the left (in space) of its corresponding cluster extrema (Poissonian particle).
Remark (Invariance property)

The law of the limiting extremal process $E = \sum_{i \in \mathbb{N}} \delta_{e_i}$ satisfies the following invariance property: For any $s \geq 0$,

$$E \overset{D}{=} \sum_{i,k} \delta_{e_i + x_k^{(i)}(s) - \sqrt{2}s},$$

where $\{x_k^{(i)}(s) : k \leq n^{(i)}(s), s \geq 0\}_{i \in \mathbb{N}}$ are i.i.d. BBMs. Indeed, use that for $t \to \infty$,

$$m(t) = m(t - s) + \sqrt{2}s + o(1).$$

Then rewrite

$$E_t \equiv \sum_{i \leq n(t)} \delta_{x_i(t) - m(t)} = \sum_{i \leq n(t - s)} \sum_{k \leq n^{(i)}(s)} \delta_{x_i(t - s) + x_k^{(i)}(s) - m(t)}$$

$$= \sum_{i \leq n(t - s)} \sum_{k \leq n^{(i)}(s)} \delta_{x_i(t - s) - m(t - s) + x_k^{(i)}(s) - \sqrt{2}s + o(1)}$$

and take $t \to \infty$. 

$m(t) = \sqrt{2}t - \frac{3}{2\sqrt{2}} \log(t)$

$E_t \equiv \sum_{i \leq n(t)} \delta_{x_i(t) - m(t)}$. 

Idea of proof of Theorem 2, [ABK12]

Theorem 2, [ABK12]
For any $0 < q < 1$, the processes $E_t^{(q)}$ converge in law to the same limit, $E^0$. Also,
\[
\lim_{r \to \infty} \lim_{t \to \infty} E_t^{(1-r/t)} = E^0.
\]
Moreover, conditionally on $Z$, $E^0 = PPP(CZ \sqrt{2}e^{-\sqrt{2}x} dx)$. (\*)

Genealogy of Extremal Particles, Theorem 2.1, [ABK11]
For any compact set $D \subset \mathbb{R}$,
\[
\lim_{r \to \infty} \sup_{t > 3r} \mathbb{P}(\exists 1 \leq k, \ell \leq n(t) : x_k(t), x_\ell(t) \in D \text{ and } d(x_k(t), x_\ell(t)) \in (r, t-r)) = 0. (**)
\]

- Same limit $E^0$ for $\frac{r}{t} < q < 1 - \frac{r}{t}$ with $r$ big enough follows from (**).
- It remains to show (\*). This is done via convergence of Laplace functionals, that is, for $\phi \in C_c^+$ we claim that
\[
\lim_{r \to \infty} \lim_{t \to \infty} \mathbb{E}\left[ e^{-\int \phi(x)E_t^{(1-r/t)}(dx)} \right] = \mathbb{E}\left[ e^{-CZ \int (1-e^{-\phi(x)}) \sqrt{2}e^{-\sqrt{2}x} dx} \right].
\]

Note. If $X \sim PPP(\lambda)$, then $\mathbb{E}[e^{-\int \phi(x)X(dx)}] = e^{\int (e^{-\phi(x)}-1)\lambda(dx)}$ (cf. [B15], Appendix).
\[ \mathcal{E}_t = \sum_{k \leq n(t)} \delta_{x_k(t) - m(t)} \text{ with } m(t) = \sqrt{2t} - \frac{3}{2\sqrt{2}} \log(t) \]

Conditional on the evolution of the BBM up to time \( r \), we obtain

\[ \mathcal{E}_t^{(1-r/t)} \overset{D}{=} \left\{ x_j(r) + M(j)(t-r) - m(t) \right\}_{j=1,\ldots,n(r)}, \]

where \( \{x^{(j)}_k(t)\}_{k \leq n^{(j)}(t)}, j \in \mathbb{N} \) are i.i.d. BBM with \( M^{(j)}(t) \equiv \max_{k \leq n^{(j)}(t)} x^{(j)}_k(t) \).

(Figure 2.4 of [ABK11])

Now,

\[ m(t) = \sqrt{2r} + m(t-r) + o(1) \text{ for } t \to \infty. \]
\[ P(M(t) - m(t) \leq x) = u(t, m(t) + x) \to w(x) \text{ unif. in } x \text{ as } t \to \infty. \]

\[ m(r) = \sqrt{2}r - \frac{3}{2\sqrt{2}} \log(r). \]

As a result,

\[
\lim_{t \to \infty} \mathbb{E} \left[ e^{-\int \phi(x) \mathcal{E}_{1-r/t}^{(1-r)}(dx)} \right] = \lim_{t \to \infty} \mathbb{E} \left[ \prod_{j=1}^{n(r)} \mathbb{E} \left[ e^{-\phi(x_j(r) - \sqrt{2}r + M(t-r) - m(t-r) + o(1))} \right] \right] = \mathbb{E} \left[ \prod_{j=1}^{n(r)} \mathbb{E} \left[ e^{-\phi(x_j(r) - \sqrt{2}r + \overline{M})} \right] \right],
\]

where \( \overline{M} \) has law \( w \). \[ \varpropto^{2} \]

**Note 1.** \( \max_{j \leq n(r)} (x_j(r) - \sqrt{2}r) \to -\infty \text{ a.s. as } r \to \infty. \)

**Note 2.** Now use asymptotics for \( \overline{M} \), i.e. \( 1 - w(x) \sim Cxe^{-\sqrt{2}x} \) for \( x \to \infty. \)
Idea of proof of Theorem 2.1, [ABK11]

Localization of Paths of Extremal Particles

**Theorem** (Genealogy of Extremal Particles, Theorem 2.1, [ABK11]) For any compact set $D \subset \mathbb{R}$,

$$
\lim_{r \to \infty} \sup_{t > 3r} \mathbb{P}(\exists 1 \leq k, \ell \leq n(t) : \bar{x}_k(t), \bar{x}_\ell(t) \in D \text{ and } d(x_k(t), x_\ell(t)) \in (r, t - r)) = 0.
$$

(Figure 2.1 of [ABK11])
For $\gamma > 0$, set
\[
  f_{t,\gamma}(s) \equiv \begin{cases} 
  s^\gamma, & 0 \leq s \leq \frac{t}{2}, \\
  (t - s)^\gamma, & \frac{t}{2} \leq s \leq t.
\end{cases}
\]

Then the upper envelope at time $t$ is defined as
\[
  U_{t,\gamma}(s) \equiv \frac{s}{t} m(t) + f_{t,\gamma}(s).
\]

**Theorem (Upper Envelope, Theorem 2.2 of [ABK11])**

Let $0 < \gamma < 1/2$. Let also $y \in \mathbb{R}$ and $\epsilon > 0$ be given. There exists $r_u = r_u(\gamma, y, \epsilon)$ such that for $r \geq r_u$ and for any $t > 3r$,
\[
  P(\exists k \leq n(t) : x_k(s) > y + U_{t,\gamma}(s) \text{ for some } s \in [r, t - r]) < \epsilon.
\]

**Idea of proof.** Discretize path and use $P(M(t) > m(t) + x) \overset{t \to \infty}{\to} 1 - w(x) \sim Cxe^{-\sqrt{2}x}$ for $x \to \infty$. 

\[\square\]
Remark (Why is this useful?)

For $\mathbb{E}[n(t)] = e^t$ i.i.d. BMs, $m(t) = \sqrt{2}t - \frac{1}{2}\frac{\log(t)}{\sqrt{2}}$ and

$$
\mathbb{E}[\#\{k \leq n(t) : B_k(t) > m(t)\}] \sim \frac{e^t}{\sqrt{2\pi t}} e^{-\frac{m(t)^2}{2t}}
$$

$$
= \frac{e^t}{\sqrt{2\pi t}} e^{-2t - \frac{2\sqrt{2}t \log(t)}{2\sqrt{2}} + \frac{\log(t)^2}{8}} = \frac{1}{\sqrt{2\pi t}} e^{\frac{\log(t)}{2} - \frac{\log(t)^2}{16t}} = \frac{1}{\sqrt{2\pi}} e^{-\frac{\log(t)^2}{16t}} = O(1).
$$

For BBM, $m(t) = \sqrt{2}t - \frac{3}{2}\frac{\log(t)}{\sqrt{2}}$. For $e^t$ i.i.d. BMs we now get

$$
\mathbb{E}[\#\{k \leq n(t) : B_k(t) > m(t)\}] = \frac{1}{\sqrt{2\pi t}} e^{3\frac{\log(t)}{2} - \frac{9\log(t)^2}{16t}} = O(t).
$$

Now, for a BM $B_t$ starting in 0,

$$
P(B_s \leq U_{t,\gamma}(s), r \leq s \leq t - r | B_t = m(t)) = P(\delta_{0,m(t)}(s) \leq U_{t,\gamma}(s), r \leq s \leq t - r) \sim \frac{1}{t}.
$$
The upper envelope can be replaced by a lower "entropic envelope" $E$:
For $\alpha > 0$ let
\[
E_{t,\alpha}(s) \equiv \frac{s}{t} m(t) - f_{t,\alpha}(s).
\]

**Theorem (Entropic Repulsion, Theorem 2.3 of [ABK11])**

Let $D \subset \mathbb{R}$ be a compact set and $0 < \alpha < 1/2$. Set $\bar{D} \equiv \sup\{x \in D\}$. For any $\epsilon > 0$ there exists $r_e = r_e(\alpha, D, \epsilon)$ such that for $r \geq r_e$ and $t > 3r$,
\[
P\left( \exists k \leq n(t) : x_k(t) \in m(t) + D, \text{ but } \exists s \in [r, t - r] : x_k(s) \geq \bar{D} + E_{t,\alpha}(s) \right) < \epsilon.
\]

(Figure 2.2 of [ABK11])
Together with a lower envelope we get with $0 < \alpha < \frac{1}{2} < \beta < 1$, 

(Figure 2.3 of [ABK11])
Remark (How to use this \( \varphi \rightarrow 3 \))

The expected number of pairs of particles of BBM whose (respective) path 
\( (x(s))_{0 \leq s \leq t} \) satisfies some conditions for \( s \in [r, t - r] \), say \( \Sigma_t^{[r, t-r]} \), is

\[
\mathbb{E}[\#\{(k, \ell) : k \neq \ell, x_k(\cdot), x_\ell(\cdot) \in \Sigma_t^{[r, t-r]}\}]
\]

\[
= C(e^t)^2 \int_0^t ds \int_{-\infty}^\infty dy \, p_s(y) \mathbb{P}(x(\cdot) \in \Sigma_t^{[r, t-r]} | x(s) = y) \mathbb{P}(x(\cdot) \in \Sigma_t^{[r \wedge s, t]} | x(s) = y).
\]

- how many pairs at time \( t \) on average;
- condition on splitting at time \( s \) and
- at position \( y \).

- If first particle satisfies the condition on \([0, t]\), then the second one automatically satisfies it on \([0, s]\).

If we include a condition on genetic distance, we get

\[
\mathbb{E}[\#\{(k, \ell) : k \neq \ell, x_k(\cdot), x_\ell(\cdot) \in \Sigma_t^{[r, t-r]}, d(x_k(t), x_\ell(t)) \in [r, t - r]\}]
\]

\[
= C e^t \int_r^{t-r} ds \int_{-\infty}^\infty dy \, p_s(y) \mathbb{P}(x(\cdot) \in \Sigma_t^{[r, t-r]} | x(s) = y) \mathbb{P}(x(\cdot) \in \Sigma_t^{[s, t-r]} | x(s) = y).
\]
ARGUIN, L.-P. and BOVIER, A. and KISTLER, N.
Genealogy of extremal particles of branching Brownian motion.

ARGUIN, L.-P. and BOVIER, A. and KISTLER, N.
Poissonian statistics in the extremal process of branching Brownian motion.

ARGUIN, L.-P. and BOVIER, A. and KISTLER, N.
The extremal process of branching Brownian motion.

AïDÉKON, E. and BERESTYCKI, J. and BRUNET, É. and SHI, Z.
Branching Brownian motion seen from its tip.

BOVIER, A.
From spin glasses to branching Brownian motion—and back?

BRAMSON, M.
Convergence of solutions of the Kolmogorov equation to travelling waves.
Chauvin, B. and Rouault, A.
KPP Equation and Supercritical Branching Brownian Motion in the Subcritical Speed Area. Application to Spatial Trees.

Gouéré, J.-B.
Branching Brownian motion seen from its leftmost particle (following Arguin-Bovier-Kistler and Aïdékon-Berestycki-Brunet-Shi).

Lalley, S. P. and Sellke, T.
A conditional limit theorem for the frontier of a branching Brownian motion.

Neveu, J.
Multiplicative martingales for spatial branching processes.

Roberts, M.I.
A simple path to asymptotics for the frontier of a branching Brownian motion.
The extremal process of BBM (cf. [ABBS13] and [ABK13])

- Arguin, L.-P. and Bovier, A. and Kistler, N.  
  The extremal process of branching Brownian motion.  

- Ådêkon, E. and Berestycki, J. and Brunet, É. and Shi, Z.  
  Branching Brownian motion seen from its tip.  

Both articles give a description of the (weak w.r.t. $\phi \in C_c^+$) limit ($t \to \infty$) of the extremal process

$$\mathcal{E}_t \equiv \sum_{k \leq n(t)} \delta_{x_k(t)} - m(t) \equiv \sum_{k \leq n(t)} \delta_{\bar{x}_k(t)}.$$

See Gouéré [G14] for a (french) review that presents and compares both approaches.
Remark

- Bovier [B15] discusses the following results in detail.
- There are also other representations. \( \rightarrow^4 \)
- The proofs rely on the consideration of the respective Laplace functionals.

Theorem (Theorem 3.1 (Existence of the limit), [ABK13])

The point process \( \mathcal{E}_t \) converges in law to a point process \( \mathcal{E} \).

Idea of proof: Example 2 for the F-KPP equation.
Theorem 2, [ABK12]

For any $0 < q < 1$, the processes $\mathcal{E}_t^{(q)}$ converge in law to the same limit, $\mathcal{E}^0$. Also,
\[
\lim_{r \to \infty} \lim_{t \to \infty} \mathcal{E}_t^{(1-r/t)} = \mathcal{E}^0.
\]
Moreover, conditionally on $Z$, $\mathcal{E}^0 = \text{PPP}(CZ \sqrt{2} e^{-\sqrt{2}x} dx)$, where $C > 0$ is the constant appearing in (5).

\[
\mathcal{E}_t \equiv \sum_{k \leq n(t)} \delta_{x_k(t)-m(t)} \quad \text{with} \quad m(t) = \sqrt{2}t - \frac{3}{2\sqrt{2}} \log(t).
\]

Definition (Cluster-extrema)

Conditionally on the limiting derivative martingale $Z$, consider the PPP
\[
\mathcal{P}_Z \equiv \sum_{i \in \mathbb{N}} \delta_{p_i} \overset{D}{=} \text{PPP}(CZ \sqrt{2} e^{-\sqrt{2}x} dx)
\]
with $C$ as in (5).

Definition (Clusters)

Let $\tilde{\mathcal{E}}_t \equiv \sum_{k \leq n(t)} \delta_{x_k(t)-\sqrt{2}t}$. Conditionally on $\{\max_{k \leq n(t)} x_k(t) - \sqrt{2}t \geq 0\}$, the process $\tilde{\mathcal{E}}_t$ converges to a point process $\tilde{\mathcal{E}} = \sum_j \delta_{\xi_j}$. Now define the point process of the gaps by
\[
\mathcal{D} \equiv \sum_j \delta_{\Delta_j}, \quad \Delta_j \equiv \xi_j - \max_j \xi_j.
\]

Note. $\mathcal{D}$ is a point process on $(-\infty, 0]$ with an atom at 0.
Theorem (Theorem 2.1 (Main Theorem), [ABK13])

Let $P_Z$ be as in (7) and let $\{D^{(i)} : i \in \mathbb{N}\}$ be a family of independent copies of the gap-process (8). Then the point process $E_t$ converges in law as $t \to \infty$ to a Poisson cluster point process $E$ given by

$$E \equiv \lim_{t \to \infty} E_t \overset{D}{=} \sum_{i,j} \delta_{p_i+\Delta^{(i)}_j}.$$

(Figure 1 of [ABK13])
[ABBS13] BBM seen from its tip

Remark

- The proofs use path localization and path decomposition techniques.
- J. Berestycki (Lecture Notes, "Topics on BBM", 
  http://www.stats.ox.ac.uk/~beresty/Articles/EBP18_v2.pdf) gives a good introduction in the underlying concepts.

Notation (Change in Scaling)

Note that a different scaling is used and instead of rightmost particles, leftmost are considered.

- Particles now follow a BM with drift 2 and variance 2, that is, replace $B_t$ by $\sqrt{2}(B_t - \sqrt{2}t)$.
- (The exponential clocks for splitting-events still ring at rate 1 and a particle splits in two.)
- Instead of $M(t) = \max_{k \leq n(t)} x_k(t)$ they consider $\min_{k \leq n(t)} x_k(t)$. 
\[ m(t) \equiv \sqrt{2}t - \frac{3}{2\sqrt{2}} \log(t). \]

The derivative mart. \[ Z(t) \equiv \sum_{k=1}^{n(t)} (\sqrt{2}t - x_k(t)) e^{-\sqrt{2}(\sqrt{2}t-x_k(t))} \] satisfies \[ Z = \lim_{t \to \infty} Z(t). \]

Particles now follow a BM with drift 2 and variance 2, that is, replace \( B_t \) by \( \sqrt{2}(B_t - \sqrt{2}t) \).

Consider \( \min_{k \leq n(t)} x_k(t) \).

Remark (Consequences of Scaling)

- \( m(t) \) becomes \( m'(t) = +\frac{3}{2} \log(t) \).
- \( Z(t) \) becomes \( \frac{1}{\sqrt{2}} \sum_{k=1}^{n(t)} x_k(t) e^{-x_k(t)} \). [ABBS13] use \( Z'(t) = \sum_{k=1}^{n(t)} x_k(t) e^{-x_k(t)} \) instead and thus \( CZ \) becomes \( (C/\sqrt{2})Z' = C'Z' \).
- \( \mathbb{E}[Z'_t] = 0 \) for all \( t \geq 0 \).

From now onwards, we use the notation of [ABBS13].

Definition (The additive martingale)

The process \( M(t) \equiv \sum_{k=1}^{n(t)} e^{-x_k(t)} \) is a martingale with \( \mathbb{E}[M_t] = 1 \) for all \( t \geq 0 \), the so-called additive martingale.
\[ E_t \equiv \sum_{k \leq n(t)} \delta_{x_k(t)-m(t)}. \]

Cluster-extrema \( P_Z \equiv \sum_{i \in \mathbb{N}} \delta_{p_i} \overset{\mathcal{D}}{=} \text{PPP}(CZ \sqrt{2} e^{-\sqrt{2}x} dx) = \text{PPP}(e^{-\sqrt{2}x+\log(CZ)} d(\sqrt{2}x)). \)

**Theorem 2.1 (Main Theorem), [ABK13]** Let \( P_Z \) be as in (7) and let \( \{\mathcal{D}(i) : i \in \mathbb{N}\} \) be a family of independent copies of the gap-process (8). Then

\[ E \equiv \lim_{t \to \infty} E_t \overset{\mathcal{D}}{=} \sum_{i,j} \delta_{p_i+\Delta_j(i)}. \]

Let

\[ \tilde{N}(t) \equiv \sum_{k \leq n(t)} \delta_{x_k(t)-m(t)+\log(CZ)}. \]

**Theorem (Theorem 2.1, [ABBS13] ⇨ 5)**

As \( t \to \infty \) the pair \( \{\tilde{N}(t), Z(t)\} \) converges jointly in distribution to \( \{L, Z\} \). \( L \) and \( Z \) are independent and \( L \) is obtained as follows.

(i) Define \( P \) a Poisson point measure on \( \mathbb{R} \), with intensity measure \( e^x dx \).

(ii) For each atom \( x \) of \( P \), we attach a point measure \( \mathcal{D}(x) \) where \( \mathcal{D}(x) \) are independent copies of a certain decoration point measure \( \mathcal{D} \).

(iii) \( L \) is then the point measure corresponding to

\[ L \equiv \sum_{x \in P} \sum_{y \in \mathcal{D}(x)} \delta_{x+y}. \]
Fig. 1 \((Y, Q)\) is the limit of the path \(s \mapsto X_{1,t}(t - s) - X_{1,t}(t)\) and of the points that have branched recently off from \(X_{1,t}\), measure corresponding to the set of all particles at time \(t\) which have branched off from \(X_{1,t}\) at time \(\tau_i(t)\) relative to the final position \(X_{1,t}(t)\) (see Fig. 1). We will also need the notation \(\tau_{i,j}(t)\) which is the time at which \(X_i(t)\) and \(X_j(t)\) share their most recent common ancestor. Observe that

\[
N_i(t) = \sum_{j \leq N(t)}: \tau_{1,j}(t) = \tau_i(t) \delta_{X_j(t) - X_{1,t}(t)}.
\]

We then define

\[
Q(t, \zeta) := \delta_0 + \sum_{i: \tau_i(t) > t - \zeta} N_i(t)
\]

i.e., the point measure of particles at time \(t\) which have branched off \(X_{1,t}\) after time \(t - \zeta\), including the particles at \(X_{1,t}(t)\) itself.

We will first show that \((Y(t), Q(t, \zeta))\) converges jointly in distribution (by first letting \(t \to \infty\) and then \(\zeta \to \infty\)) towards a limit \((Y(s), Q)\) where the second coordinate is our point measure \(Q\) which is described by growing conditioned branching Brownian motions born at a certain rate on the path \(Y\). We first describe the limit \((Y(s), Q)\) and then we state the precise convergence result.

The following family of processes indexed by a real parameter \(b > 0\) play a key role in this description. Let \(B := (B_t, t \geq 0)\) be a standard Brownian motion and let \(R := (R_t, t \geq 0)\) be a three-dimensional Bessel process started from \(R_0 = 0\) and independent from \(B\) (see Fig. 2). Let us define

\[
T_b := \inf \{ t \geq 0 : B_t = b \}.
\]

For each \(b > 0\), we define the process \(\Gamma_1(b)\) as follows:

\[
\Gamma_1(b)(s) :=
\begin{cases}
  B_s, & \text{if } s \in [0, T_b], \\
  b - R_s - T_b, & \text{if } s \geq T_b.
\end{cases}
\]

(Figure 1 of [ABBS13])
Notation

- **Order the particles in increasing order**, i.e. \( x_1(t) \leq x_2(t) \leq \cdots \leq x_{n(t)}(t) \).
- For \( s \leq t \), let \( x_{1,t}(s) \) denote the position at time \( s \) of the ancestor of \( x_1(t) \), i.e. \( s \mapsto x_{1,t}(s) \) is the path of the leftmost particle up until time \( t \).
- Let \( Y_t(s) \equiv x_{1,t}(t - s) - x_1(t), \quad s \in [0, t] \) the **time reversed path back from the final position** \( x_1(t) \).
- Denote by \( \tau_2(t) < \tau_1(t) \leq t \) the **successive splitting times** along the path of the leftmost particle (enumerated backwards).
- The time at which \( x_i(t) \) and \( x_j(t) \) share their MRCA is denoted by \( \tau_{i,j}(t) \).
- Let \( N_i(t) \equiv \sum_{1 \leq j \leq n(t): \tau_{1,j}(t) = \tau_i(t)} \delta x_j(t) - x_1(t) \).
- Finally, let for \( 0 < \eta < t \),
  \[
  D(t, \eta) \equiv \delta_0 + \sum_{i: \tau_i(t) > t - \eta} N_i(t).
  \]
Theorem (Theorem 2.3, [ABBS13])

The following convergence holds jointly in distribution:

\[ \lim_{\eta \to \infty} \lim_{t \to \infty} ((Y_t(s), s \in [0, t]), \mathcal{D}(t, \eta), x_1(t) - m(t)) = ((Y(s), s \geq 0), \mathcal{D}, W), \]

where the r.v. \( W \) is independent of the pair \( ((Y(s), s \geq 0), \mathcal{D}) \), and \( \mathcal{D} \) is the point measure which appears in Theorem 2.1.

Note. \( \mathbb{P}(W(x) \leq x) = 1 - w(-x/\sqrt{2}) \sim C'|x|e^x \) for \( x \to -\infty \) (cf. (5) and below).
Construction of the decoration point measure $\mathcal{D}$

We will construct $\mathcal{D}$ conditional on $Y$.

Notation

Let $b > 0$, $(B_t, t \geq 0)$ a BM and $(R_t, t \geq 0)$ a three-dimensional Bessel process started from $R_0 = 0$ and independent of $B$. Let $T_b \equiv \inf\{t \geq 0 : B_t = b\}$. Set

$$\Gamma_s^{(b)} \equiv \begin{cases} B_s, & s \in [0, T_b], \\ b - R_{s-T_b}, & s \geq T_b. \end{cases}$$

(Figure 1 and 2 of [ABBS13])
Construction of $\mathcal{D}$

(1) **Construction of $Y$.**

For $A \subset \mathcal{C}(\mathbb{R}^+, \mathbb{R})$ measurable,

$$
P(Y \in A, -\inf_{s \geq 0} Y(s) \in db) = \frac{1}{c} \mathbb{E} \left[ e^{-2 \int_0^\infty P(x_1(v) \leq \sqrt{2} \Gamma_v(b)) \, dv} 1_{\{ -\sqrt{2} \Gamma(b) \in A \}} \right]
$$

with normalizing constant $c$.

(2) **Construction of $\mathcal{D}$ conditional on $Y$.**

Conditionally on the path $Y$, let $\pi$ be a PPP on $[0, \infty)$ with intensity $2 \cdot P(Y(\tau) + x_1(\tau) > 0) d\tau$. For each point $\tau \in \pi$ start an independent BBM $(\mathcal{N}_{Y(\tau)}^*(u), u \geq 0)$ at position $Y(\tau)$ **conditioned** to have $\min \mathcal{N}_{Y(\tau)}^*(\tau) > 0$. Then

$$
\mathcal{D} \equiv \delta_0 + \sum_{\tau \in \pi} \mathcal{N}_{Y(\tau)}^*(\tau).
$$

(Recall that $Y(0) = 0$ and that the path $Y$ moves backwards in time, whereas the BBMs move forward in time.)
Spinal decomposition

The process $M(t) \equiv \sum_{k=1}^{n(t)} e^{-x_k(t)}$ is a martingale with $\mathbb{E}[M_t] = 1$ for all $t \geq 0$, the so-called additive martingale.

Let $Q$ be the probability measure s.t.

$$Q|_{\mathcal{F}_t} = M(t) \cdot P|_{\mathcal{F}_t},$$

where $P$ refers to the distribution of BBM and $\mathcal{F}_t$ is the filtration of the BBM (under $P$) up to time $t$.

**Theorem (Theorem 5 of Chauvin and Rouault [CR88])**

$Q$ is the law of the following branching diffusion.

1. Let $\Xi_s \in \{1, \ldots, n(s)\}$ denote the label of a distinguished particle at time $s \geq 0$ with

$$Q(\Xi_t = i|\mathcal{F}_t) = \frac{e^{-x_i(t)}}{M_t}.$$

The process $(\Xi_s, s \in [0, t])$ is called the spine.

2. The position of the spine $(x_{\Xi_s}(s), s \in [0, t])$ is a driftless BM of variance 2.

3. The particle with label $\Xi_s$ at time $s$ branches at (accelerated) rate 2 and gives birth to BBMs (with distribution $P$).
Lemma (Many-to-one-principle)

Let $\Psi_i, i = 1, \ldots, n(t)$ be $\mathcal{F}_t$-measurable r.v.s. Then

$$
\mathbb{E}_\mathbb{P}\left[ \sum_{i \leq n(t)} \Psi_i \right] = \mathbb{E}_\mathbb{Q}\left[ \frac{1}{\mathcal{M}(t)} \sum_{i \leq n(t)} \Psi_i \right] = \mathbb{E}_\mathbb{Q}[e^{\Xi_t(t)} \Psi_{\Xi_t}].
$$

Example

We obtain

$$
\mathbb{P}(\exists i \leq n(t) : (x_{i,t}(s), s \in [0, t]) \in A) \leq \mathbb{E}\left[ \sum_{i \leq n(t)} 1\{x_{i,t}(s), s \in [0, t]) \in A\} \right]
$$

$$
= \mathbb{E}\left[ e^{\Xi_t(t)} 1\{(\sqrt{2}B_s, s \in [0, t]) \in A\} \right]
$$

$$
= \mathbb{E}\left[ e^{\sqrt{2}B_t} 1\{((\sqrt{2}B_s, s \in [0, t]) \in A\} \right].
$$
A note on the i.i.d. case

(cf. A. Bovier, Lecture Notes, "Extreme values of random processes",
https://wt.iam.uni-bonn.de/fileadmin/WT/Inhalt/people/Anton_Bovier/lecture-notes/extreme.pdf, Lemma 1.2.1)

Let \( X_1(t), \ldots, X_n(t) \) be \( n \in \mathbb{N} \) i.i.d. normal r.v.s. Let

\[
b_n \equiv \sqrt{2 \log(n)} - \frac{\log(\log(n)) + \log(4\pi)}{2\sqrt{2 \log(n)}} \quad \text{and} \quad a_n \equiv \sqrt{2 \log(n)}.
\]

Then, for all \( x \in \mathbb{R} \),

\[
\lim_{n \to \infty} \mathbb{P}(\max_{k \leq n} X_k(t) \leq b_n + x/a_n) = e^{-e^{-x}}.
\]

For a BBM, \( \mathbb{E}[n(t)] = e^t \). Set \( n = e^t \) and consider \( B_i, i \in \mathbb{N} \) independent standard BMs. Then, for \( y \equiv x/\sqrt{2} \),

\[
\lim_{n \to \infty} \mathbb{P}\left(\max_{k \leq e^t} \frac{B_k(t)}{\sqrt{t}} \leq \sqrt{2t} - \frac{\log(t)}{2\sqrt{2t}} + \frac{\log(4\pi) + 2x}{2\sqrt{2t}}\right) = \lim_{t \to \infty} \mathbb{P}\left(\max_{k \leq e^t} B_k(t) \leq \sqrt{2t} - \frac{1}{2\sqrt{2}} \cdot \log(t) + O(1) + y\right) = e^{-e^{-\sqrt{2}y}}.
\]

Recall, that for BBM,

\[
m(t) = \sqrt{2t} - \frac{3}{2\sqrt{2}} \log(t) \quad \text{and} \quad \mathbb{P}(M(t) - m(t) \leq x) \xrightarrow{t \to \infty} \mathbb{E}[e^{-CZe^{-\sqrt{2}x}}].
\]
Idea of proof of Theorem 2, [ABK12] - Heuristic

1 − w(x) ∼ Cxe^{−\sqrt{2}x} for x → ∞. Suppose heuristically w′(x) ∼ C\sqrt{2}xe^{−\sqrt{2}x}.

Z(t) = ∑_{k=1}^{n(t)} (\sqrt{2}t − x_k(t)) e^{−\sqrt{2}(\sqrt{2}t−x_k(t))} → Z for t → ∞.

\[
\lim_{t \to \infty} \mathbb{E} \left[ e^{-\int \phi(x) \mathcal{E}_t^{1-r/t}(dx)} \right] = \mathbb{E} \left[ \prod_{j=1}^{n(r)} \mathbb{E} \left[ e^{-\phi(x_j(r) - \sqrt{2}r + \bar{M})} \right] \right],
\]

where \( \bar{M} \) has law \( w \) and \( \max_{j \leq n(r)} (x_j(r) - \sqrt{2}r) \to -\infty \) a.s. as \( r \to \infty \).

Rewrite the above to \((\log(ab) = \log(a) + \log(b), \log(x) \sim -(1 − x) \text{ for } 0 < x < 1)\)

\[
\mathbb{E} \left[ \sum_{j=1}^{n(r)} \log \left( \mathbb{E} \left[ e^{-\phi(x_j(r) - \sqrt{2}r + \bar{M})} \right] \right) \right] \sim \mathbb{E} \left[ e^{-\sum_{j=1}^{n(r)} \mathbb{E} \left[ 1 - e^{-\phi(x_j(r) - \sqrt{2}r + \bar{M})} \right]} \right]
\]

\[
\sim \mathbb{E} \left[ e^{-\sum_{j=1}^{n(r)} \int (1-e^{-\phi(x)}) \mathbb{P} (\bar{M} = -x_j(r) + \sqrt{2}r + dx)} \right]
\]

\[
\sim \mathbb{E} \left[ e^{- \int (1-e^{-\phi(x)}) \sum_{j=1}^{n(r)} C \sqrt{2} \left( -x_j(r) + \sqrt{2}r + x \right) e^{-\sqrt{2} \left( -x_j(r) + \sqrt{2}r + x \right)} dx} \right]
\]

\[
r \to \infty \quad \mathbb{E} \left[ e^{- \int C \sqrt{2} \left( 1-e^{-\phi(x)} \right) Z e^{-\sqrt{2}x} dx} \right].
\]

- Let $k, \ell$ be such that $d(x_k(t), x_\ell(t)) = s \in [r, t - r]$. Consider the case $s \leq t/2$ and to simplify calculations $s = O(t)$ in what follows.

Due to entropic repulsion: w.l.o.g.

$$x_k(t), x_\ell(t) \in m(t) + D$$

and

$$x_k(s) = x_\ell(s) < \bar{D} + E_{t,\alpha}(s) = \bar{D} + \sqrt{2} s - \frac{s}{t} \frac{3}{2 \sqrt{2}} \log(t) - s^\alpha$$

$$\sim \sqrt{2} s - s^\alpha$$

for some fixed $0 < \alpha < 1/2$ and for all $s \in [r, t - r]$.

- For particles $k$ and $\ell$ to reach $m(t) + D$, the MRCA of $k$ and $l$ (at time $s$) must itself produce a BBM that after a time-interval of length $t - s$ has height at least

$$\left( m(t) - \min(D) \right) - \left( \sqrt{2} s - s^\alpha \right) \sim \sqrt{2} (t - s) + s^\alpha.$$
• **Number of possible choices for ancestor:** At time $s$, there are on average $e^s$ particles. The chance of one particle (i.e., BM) to reach $(x_k(s) =) \sqrt{2}s - s^{\alpha}$ is of order
\[
\frac{1}{\sqrt{2\pi} s} e^{-\frac{2s^2 - 2\sqrt{2}ss^{\alpha} + s^{2\alpha}}{2s}} = \frac{1}{\sqrt{2\pi} s} e^{-s} e^{\sqrt{2}s^{\alpha}} e^{-\frac{s^{2\alpha} - 1}{2}},
\]
where $2\alpha - 1 < 0$. In the product (more particles at this height as if we consider BBM) at time $s$ we have on average at most of order $e^{\sqrt{2}s^{\alpha}}$ choices.

• **Chance for ancestor (at time $s$) to have a child at height $\sqrt{2}(t - s) + s^{\alpha}$ (at time $t$):** Starting with a single particle, the probability that BBM jumps this high in the time-interval $t - s$ is (use that $t - s \geq r \gg$ and $s = O(t)$ and $\mathbb{P}(M(t) - m(t) > x) = 1 - w(x) \sim C xe^{-\sqrt{2}x}$ for $x \to \infty$)
\[
\mathbb{P}(M(t - s) \geq \sqrt{2}(t - s) + s^{\alpha})
= \mathbb{P}\left(M(t - s) - m(t - s) \geq \frac{3}{2\sqrt{2}} \log(t - s) + s^{\alpha}\right)
\sim 1 - w\left(\frac{3}{2\sqrt{2}} \log(t - s) + s^{\alpha}\right) \sim C(s^{\alpha} + \delta)e^{-\sqrt{2}(s^{\alpha} + \delta)} \sim e^{-\sqrt{2}s^{\alpha}}.
\]

• **Chance to have two (that split immediately):** of order $(e^{-\sqrt{2}s^{\alpha}})^2$.

• **Overall chance:** of order $e^{-\sqrt{2}s^{\alpha}}$, so negligible.
Other Representations for the extremal process of BBM, (cf. [ABK13])

Theorem 2, [ABK12]
For any $0 < q < 1$, the processes $\mathcal{E}_t^{(q)}$ converge in law to the same limit, $\mathcal{E}^0$. Also, 
\[ \lim_{r \to \infty} \lim_{t \to \infty} \mathcal{E}_t^{(1-r/t)} = \mathcal{E}^0. \] Moreover, conditionally on $Z$, $\mathcal{E}^0 = PPP(CZ\sqrt{2}e^{-\sqrt{2}x}dx)$.

Proposition (Proposition 3.2, [ABK13])
For $\phi \in C_c^+(\mathbb{R})$ and any $x \in \mathbb{R}$,
\[ \lim_{t \to \infty} \mathbb{E}\left[ e^{-\int \phi(y+x)\mathcal{E}_t(\,dy)} \right] = \mathbb{E}\left[ e^{-C(\phi)Ze^{-\sqrt{2}x}} \right] \]
where, for $v(t, y)$ the solution of F-KPP with initial condition $v(0, y) = e^{-\phi(y)}$,
\[ C(\phi) = \lim_{t \to \infty} \sqrt{\frac{2}{\pi}} \int_0^\infty (1 - v(t, y + \sqrt{2}t))ye^{\sqrt{2}y} dy \]
is a strictly positive constant depending on $\phi$ only.
\[ \mathcal{P}_Z \equiv \sum_{i \in \mathbb{N}} \delta_{p_i} \overset{\mathcal{D}}{=} \text{PPP}(CZ\sqrt{2}e^{-\sqrt{2}x}dx) \]

Let \( \tilde{\mathcal{E}}_t \equiv \sum_{k \leq n(t)} \delta_{x_k(t)-\sqrt{2}t} \). \textbf{Conditionally on} \{ \max_{k \leq n(t)} x_k(t) - \sqrt{2}t \geq 0 \}, the process \( \tilde{\mathcal{E}}_t \) converges to a point process \( \tilde{\mathcal{E}} = \sum_j \delta_{\xi_j} \). Now define the \textbf{point process of the gaps} by

\[ \mathcal{D} \equiv \sum_j \delta_{\Delta_j}, \quad \Delta_j \equiv \xi_j - \max_j \xi_j. \]

\textbf{Note.} \( \mathcal{D} \) is a point process on \( (-\infty, 0] \) with an atom at 0.

\textbf{Theorem 2.1 (Main Theorem)}, [ABK13] Let \( \mathcal{P}_Z \) be as in (7) and let \( \{\mathcal{D}(i) : i \in \mathbb{N}\} \) be a family of independent copies of the gap-process (8). Then

\[ \mathcal{E} \equiv \lim_{t \to \infty} \mathcal{E}_t \overset{\mathcal{D}}{=} \sum_{i,j} \delta_{p_i+\Delta_j(i)}. \]

Let \( (\eta_i : i \in \mathbb{N}) \) be the atoms of a PPP on \( (-\infty, 0) \) with intensity measure

\[ \sqrt{\frac{2}{\pi}} (-x)e^{-\sqrt{2}x}dx. \]

For each \( i \in \mathbb{N} \) consider independent BBMs with drift \(-\sqrt{2}\), i.e. \( \{x_{k}^{(i)}(t) - \sqrt{2}t : k \leq n^{(i)}(t)\} \). The \textbf{auxiliary point process} is defined as

\[ \Pi_t \equiv \sum_{i,k} \delta \frac{1}{\sqrt{2}} \log(Z)+\eta_i+x_{k}^{(i)}(t)-\sqrt{2}t. \]

\textbf{Theorem (Theorem 3.6 (The auxiliary point process)}, [ABK13])

\[ \mathcal{E} \overset{\mathcal{D}}{=} \lim_{t \to \infty} \Pi_t. \]
Heuristic for appearance of the Poisson point measure

(Proposition 10.1 in [ABBS13])

Fix \( k \geq 1 \).

- Let \( \mathcal{H}_k \) be the set of particles (at position \( k \)) that hit the spatial position \( k \) first in their line of descent.

  **Note:** Conditionally on \( \mathcal{H}_k \), the subtrees rooted at the points of \( \mathcal{H}_k \) are independent BBMs started at position \( k \) and at a random time (i.e. when the particle of \( \mathcal{H}_k \) hit \( k \)).

- Define \( H_k \equiv \# \mathcal{H}_k \). **Note:** finite a.s. (use \( m(t) = + \frac{3}{2} \log(t) \) and that we consider the minimum of BBM)

- Now let

  \[
  Z_k \equiv ke^{-k}H_k.
  \]

\[
Z = \lim_{t \to \infty} Z(t) = \lim_{t \to \infty} \sum_{k=1}^{n(t)} x_k(t) e^{-x_k(t)}.
\]

Neveu ([N88], (5.4)) shows that

\[
\lim_{k \to \infty} Z_k = Z, \text{ a.s.}
\]
• Let \( \mathcal{H}_{k,t} \subset \mathcal{H}_k \) be the set of all particles that hit \( k \) before time \( t \).

• For \( u \in \mathcal{H}_{k,t} \), write \( x_1^u(t) \) for the minimal position at time \( t \) of the particles which are descendants of \( u \).

   If \( u \in \mathcal{H}_k \setminus \mathcal{H}_{k,t} \), let \( x_1^u(t) = 0 \).

Now define the point measures

\[
P_{k,t}^* = \sum_{u \in \mathcal{H}_k} \delta_{x_1^u(t) - m(t) + \log(CZ_k)}
\]

and \( (Z_k \equiv k e^{-k} H_k \text{ and } m(t) = \frac{3}{2} \log(t), \text{ i.e. } m(t + c) - m(t) \to 0 \text{ for } t \to \infty) \)

\[
P_{k,\infty}^* = \sum_{u \in \mathcal{H}_k} \delta_{k + W(u) + \log(CZ_k)},
\]

where, conditionally on \( \mathcal{F}_{\mathcal{H}_k} \) (sigma-algebra generated by the BBM when the particles are stopped upon hitting the position \( k \)), the \( W(u) \) are independent copies of the r.v. \( W \).

Proposition (Proposition 10.1 of [ABBS13])

The following convergences hold in distribution.

\[
\lim_{t \to \infty} P_{k,t}^* = P_{k,\infty}^* \quad \text{and} \quad \lim_{k \to \infty} (P_{k,\infty}^*, Z_k) = (P, Z)
\]

where \( P \) is as in Theorem 2.1 and \( P \) and \( Z \) are independent.