On the Applicability of Convex Relaxations for Matching Symmetric Shapes

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Shape matching

Finding isometries
Shape matching-finding isometries

\[ A_{ij} = d(a_i, a_j) \]

\[ B_{kl} = d(b_k, b_l) \]

Goal: find mapping (permutation)
\[ \sigma: \{a_1, ..., a_n\} \rightarrow \{b_1, ..., b_n\} \]
such that
\[ A_{ij} = B_{\sigma(i), \sigma(j)} \]
**Graph isomorphism**

**Input:** $A = A^T, B = B^T$

**Goal:** find permutation (if exists) $\sigma$ such that $A_{ij} = B_{\sigma(i),\sigma(j)}$

**Goal:** find permutation matrix $P \in \Pi_n$ such that $A = P^TBP$
Graph matching/quadratic assignment

Input: \( A = A^T, B = B^T \)

Output: \( P \in \Pi_n \) such that \( A \approx P^T B P \):

\[
P^* = \arg\min_{P \in \Pi_n} \| A - P^T B P \|_F
\]

Graph matching
DS relaxation

\[
P^* = \text{argmin}_{P \in \Pi_n} \|PA - BP\|_F
\]

\[
S^* = \text{argmin}_{S \in \text{conv} \Pi_n} \|SA - BS\|_F
\]

NP Hard!
**DS relaxation**

conv $\Pi_n$ is the Birkhorff polytope:

$$DS = \{S \mid S \succeq 0, S1 = 1, S^T 1 = 1\}$$

**DS relaxation:**

$$S^* = \text{arg} \min_{S \in DS} \|SA - BS\|_F$$

$$GM_{DS}(A, B)$$
DS relaxation—does it work?

\[ P^* = \text{argmin}_{P \in \Pi_n} ||PA - BP||_F \]

Part I: exactness

Part II: projection

\[ S^* = \text{argmin}_{S \in DS} ||SA - BS||_F \]
Exactness affects projection efficiency

\[ S^* = \arg\min_{S \in D_S} ||SA - BS||_F \]
Part I: Exactness

Assume: (i) $A \cong B$ (ii) Unique isomorphism $P^*$

$$\min_{S \in DS} ||AS - SB||_F = 0$$

$$S^* = \arg\min_{S \in DS} ||SA - BS||_F$$
Problem: Unique solution assumption

ψ, ψ ∘ φ

Symmetric
Symmetries of natural shapes

Bilateral symmetry: [SCAPE, FAUST, TOSCA]  [SHREC]
Exactness vs. convex exactness

Exactness (asymmetric)

Convex exactness (symmetric)
Convex exactness-definition

\( Iso(A, B) = \{ P \in \Pi_n \mid AP = PB \} \), \( Iso_{conv}(A, B) = \{ S \in DS \mid AS = SB \} \)

\( GM_{DS}(A, B) \) is convex exact if

\[
Iso_{conv}(A, B) = conv(Iso(A, B))
\]
A convenient reduction (B=A)

(easy) Lemma:

\[ GM_{DS}(A, A) \text{ is convex exact} \]

For any \( B \) s.t. \( B \cong A \),
\[ GM_{DS}(A, B) \text{ is convex exact} \]
Goal

\[ P_1^* \times P_2^* \times S^* \]

Almost surely

Usually

Almost surely

Usually?

Usually not

Usually not?
Measure for the space of asymmetric graphs

Asymmetric graphs: \( \{ A = A^T \mid A \text{ has no non-trivial automorphisms} \} \)

\( S^n = \{ A \mid A = A^T \} \)

Asymmetric graphs

\[ V(G) = \{ A \in S^n \mid PA = AP, \forall P \in G \} \]
Measure for Graphs with prescribed symmetry group $G_0$

Graphs with sym group $G_0$: Graphs whose automorphism group is $G_0$

\[
V(G_0) = \{A \in S^n | PA = AP, \forall P \in G_0\}
\]
Measure for Graphs with prescribed symmetry group $G_0$

$V(G_0)$

$\mu_{G_0} = \text{Lebesgue}$

Graphs with sym group $G_0$

$V(G_1)$

$V(G_2)$

$V(G_3)$

$G_i > G_0$

$V(G) = \{ A \in S^n | PA = AP, \forall P \in G \}$
Convex exactness for reflective groups

Theorem 1: If $G \cong Z_2$, then for $\mu_G$ almost every $A$, $GM_{DS}(A, A)$ is convex exact.

Also true for $G = \{I_n\}$
In general Theorem 1 holds if

- $G$ is reflective ($P^2 = I_n, \forall P \in G$).
- $G$’s action on the vertices has a full orbit.
General groups: 0-1 probability

Theorem 2: For any $G \leq \Pi_n$, either

(i) For $\mu_G$ a.e. $A$, $GM_{DS}(A, A)$ is convex exact.

Or

(ii) For all $A$ with sym group $G$, $GM_{DS}(A, A)$ is not convex exact.

Proof is constructive...
\[ A_{ij} = \| p_i - p_j \|_2 \]

\[ G_{MDS}(A, A) \]

\[ A_{ij} = \| p_i - p_j \|_2 \]

\[ G_{MDS}(A, A) \]

\[ A_{ij} = \| p_i - p_j \|_1 \]

\[ G_{MDS}(A, A) \]
General groups: 0-1 probability

Theorem 2: For any $G \leq \Pi_n$, either

(i) For $\mu_G$ a.e. $A$, $GM_{DS}(A, A)$ is convex exact.
    Or

(ii) For all $A$ with sym group $G$, $GM_{DS}(A, A)$ is not convex exact.
Summary - Part I convex exactness

- almost everywhere
- ? almost everywhere
Part II: Where’s my permutation?

set of minimizers = \{S \in DS | AS = SB\}

Simplex algorithm
Part II: Where’s my permutation?
DS relaxation- $L_2$ projection

\[ S^* = \arg\min_{S \in DS} ||SA - BS||_F \]

\[ P_{L_2} = \arg\min_{P \in \Pi} ||S^* - P||_F \]
DS++: convex2concave projection
convex2concave projection

[Zaslavskiy, Bach and Vert 2009]

Observation: Convex energy $E_0$ is equivalent over $\Pi_n$ to concave energy $E_T$.

**Concave energy:**
1. Local/global minima are permutations!
2. Intractable

**Convex energy:**
1. Minima may not be permutations
2. Tractable!

![Diagram showing local and global minima for concave and convex energies](image)
convex2concave projection

\[
S_k = \text{"argmin"}_{S \in D} E_{t_k}(S)
\]

Warm start optimization from \(S_{k-1}\).
DS++: convex2concave projection

\[ E_t(S) = \|SA - BS\|_F^2 + t(n - \|S\|_F^2) \]

- \( E_0 = E \)
- \( E_t = E \) over \( \Pi_n \)
- \( E_{t_F} \) strictly concave for \( t_F \gg 0 \)

\[ n - \|S\|_F^2 = 0 \]
Choosing \([t_0, t_F]\)

\[ E_t(S) = \|SA - BS\|_F^2 + t(n - \|S\|_F^2) \]

Best choice of \(t_F\): \(t_F = \lambda_{\text{max}}\)

\(t_F = \lambda_{\text{max}}\) over \(V_{DS} = \{S | S1 = 0, S^T 1 = 0\}\)

Best choice of \(t_0\):

\((\text{DS})\) \(t_0 = 0\) \[\text{[Aflalo et al. 15]}\]

\((\text{DS+})\) \(t_0 = \lambda_{\text{min}}\) \[\text{[Fogel et al. 13,15]}\]

\((\text{DS++})\) \(t_0 = \lambda_{\text{min}}\) over \(V_{DS}\)
Relaxation comparison
DS++ vs local minimization

DS++ vs local minimization with 1000 different initializations:

![Graph comparing DS++ and local minimization with 1000 initializations. The x-axis represents experiment number, the y-axis represents objective value. The graph shows the performance of DS++ (blue line) and Solomon 16 (red line).]
Projection comparison

![Graph showing retrieval ratio vs. noise (10^x)]

- **Retrieval ratio**
- **Noise (10^x)**

Lines: $L_2$, DS++
Symmetric, no noise

\[ E_t(S) = ||SA - BS||_F^2 + t(n - ||S||_F^2) \]

For \( t > 0 \), \( P_i^* \) are the only global minima!
Symmetric, no noise

Theorem 3: If DS(A,B) is convex exact, then (under some conditions)
\[ S_1, S_2, \ldots, S_T = P_i^* \]
Thank you!

For more details see:
“Exact Recovery with Symmetries for the Doubly-Stochastic Relaxation.”
“DS++: A Flexible, Scalable and Provably Tight Relaxation for Matching Problems.”

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