Applications of Conformal Geometry in Brain Mapping & Computational Anatomy

Dr. Monica K. Hurdal
Department of Mathematics
Florida State University, Tallahassee, U.S.A.

mhurdal@math.fsu.edu
http://www.math.fsu.edu/~mhurdal
The Human Brain
Cortical Flat Maps of the Brain

• Functional processing mainly on cortical surface
• 2D analysis methods desired: **Cortical Flat Maps**
• Metric-based approaches (i.e. area or length preserving maps) will always have distortion
• Conformal maps offer a number of useful properties including:
  – mathematically unique
  – different geometries available
  – canonical coordinate system
Potential Advantages of Brain Flat Maps

• Cortical flat maps facilitate the determination and analysis of spatial relationships between different cortical regions

• Definition of coordinate system on cortical surface

• Comparison of individual differences in cortical organization or in functional foci
  – identify/quantify specific regions where diseases occur
  – analysis of regions buried within sulci

• Visualization of cortical folding patterns
“Flattening” Surfaces and Conformal Mapping

• By a “flat” surface, we mean a surface of constant curvature:
  – Euclidean plane (identified with the complex plane),
    \[ R^2 = \mathbb{C} = \{ z = x + iy : x, y \in \mathbb{R} \} \]
  – the unit disc in \( \mathbb{C} = \mathbb{D} = \{ (x,y) : x^2 + y^2 < 1 \} \)
  – the unit sphere \( \mathbb{S} = \{ (x,y,z) : x^2 + y^2 + z^2 = 1 \} \subset \mathbb{R}^3 \)

• Why Conformal? Impossible to flatten a surface with intrinsic curvature without introducing metric or areal distortions: “Map Maker’s Problem”
  BUT we can preserve angles \( \Rightarrow \) Conformal Maps
Uniformization Theorem (1880s)

- Generalization of the Riemann Mapping Theorem (1851)

There exists a unique conformal mapping (up to conformal automorphisms) from a Riemann surface to the Euclidean plane, hyperbolic disc or sphere.

Conformal Maps Exist
and are
Mathematically Unique!
Discrete Conformal Mapping

Given a triangulated mesh: angle sum $\Theta$ at a vertex $v$ is sum of angles from triangles emanating out of $v$.

The angle at a vertex in original surface maps such that it is the Euclidean measure rescaled so the total angle sum measure is $2\pi$.

In the discrete setting, this corresponds to preserving angle proportion: an angle $\theta_i$ at a vertex $v$ in original surface has angle $2\pi\theta_i / \Theta$ in mapping to the Euclidean plane.
Conformal Mapping Methods

**Numerical Methods**

- PDE methods for solving Cauchy-Riemann equations
- Harmonic energy minimization for solving Laplace-Beltrami equation
- Differential geometric methods based on approximation of holomorphic differentials

**Circle Packing Method**

- Circle packing computed to find conformal map
Discrete Conformal Mapping with Circle Packings

A circle packing is a configuration of circles with a specified pattern of tangencies.

Theoretical, computational developments use circle packings to approximate a conformal mapping.

Circle Packing Theorem & Ring Lemma guarantee this circle packing is unique and quasi-conformal.

Collaboration with Ken Stephenson, Mathematics, U. Tennessee, Knoxville
Given a simply-connected triangulated surface:
- represent each vertex by a circle such that each vertex is located at the center of its circle
- if two vertices form an edge in the triangulation, then require their corresponding circles must be tangent in the final packing
- assign a positive number to each boundary vertex
The (Euclidean) Algorithm

- Iterative algorithm has been proven to converge
- Surface curvature is concentrated at the vertices
- A set of circles can be “flattened” in the plane if the angle sum around a vertex is $2\pi$
- To “flatten” a surface at the interior vertices:

  Positive curvature or cone point  
  (angle sum $< 2\pi$)  

  Zero curvature  
  (angle sum $= 2\pi$)  

  Negative curvature or saddle point  
  (angle sum $> 2\pi$)
For all faces $\langle v,u,w \rangle$ containing vertex $v$:

$$\sum_{\langle v,u,w \rangle} \arccos\left\{ \frac{(r_v + r_u)^2 + (r_v + r_w)^2 - (r_u + r_w)^2}{2(r_v + r_u)(r_v + r_w)} \right\} = 2\pi$$
The Algorithm (Continued)

- This collection of tangent circles is a circle packing and gives a new surface in $\mathbb{R}^2$ which is our quasi-conformal flat mapping.
- Easy to compute the location of the circle centers in $\mathbb{R}^2$ once the first 2 tangent circles are laid out.
- Each circle in the flat map corresponds to a vertex in the original 3D surface.
- Similar algorithm exists for hyperbolic geometry.
- No known spherical algorithm: use stereographic projection to generate spherical map.
- **NOTE:** A packing only exists once all the radii have been computed!
- Theorem (Bowers-Stephenson): This scheme converges to a conformal picture of the triangulation with repeated hexagonal refinement of the triangulation and repacking.
Creating a Cortical Surface

• MRI volume stripped of extraneous regions (i.e. scalp, skull, csf) to leave the region of interest (ROI)

• Resulting volume smoothed and a surface reconstruction algorithm, such as marching cubes applied to produce a triangulated mesh representing the surface of the brain

• The human brain is topologically equivalent to an orientable, 2-manifold (ie. a sphere)

• A boundary may be introduced by introducing cuts to make the brain topologically equivalent to a closed disc

• Problem: Many surface reconstruction algorithms produce a surface with topological problems - these must be fixed

• If a surface is topologically correct, then it is a topological sphere if and only if Euler characteristic \( = v - e + f = 2 \)
Topological Surface Problems

- **Unused vertices**: vertices not forming a triangle
- **Duplicate triangles**: repeated triangles
- **Edge problems**: walls, ridges, bubbles, holes
- **Surface connectedness**: only one surface
- **Triangle orientation**: all counter clockwise
- **Vertex singularity**: pinched surface
- **Handles**
  - very difficult problem
  - each handle contributes -2 to the Euler characteristic
  - need to find handle: cut and cap ends or fill
Magnetic Resonance Imaging (MRI)

Axial Slice

Coronal Slice

Sagittal Slice
Reconstruction of MR Images
Neural Tissue Reconstruction
Mapping a Cortical Hemisphere

181,154 vertices
362,304 triangles
Mapping to the Plane

170,909 vertices
341,463 triangles

Euclidean Map

Hyperbolic Map
Visualizing Flat Maps
Mapping a Cerebellum
Data courtesy of D. Rottenberg, U. Minnesota
Euclidean & Spherical Maps

Surface:
28,340 vertices
56,676 triangles
Hyperbolic Maps
Flat Maps of Different Subjects
Twin Study

Collaboration with Center for Imaging Science (Biomed. Eng.), Johns Hopkins U. & Psychiatry and Radiology Departments, Washington U. School of Medicine

- As with non-twin brains, identical twin brains have individual variability i.e. brains are *NOT* identical in the location, size and extent of folds
- Aim: determine if twin brains more similar than non-twin brains
  - if so, this can be used to help identify where a disease manifests itself if one twin has a disease/condition that the other does not
- Flat maps can help identify similarities and differences in the curvature and folding patterns
- Examining ventral medial prefrontal cortex (VMPFC)
CS: Cingulate Sulcus
GR: Gyrus Rectus
IRS: Inferior Rostral Sulcus
OFC: Olfactory Cortex
VMPFC: Ventral Medial Prefrontal Cortex

OFS: Olfactory Sulcus
ORS: Orbital Sulcus
PS: Pericallosal Sulcus
SRS: Superior Rostral Sulcus
CG = cingulate gyrus
CS = cingulate sulcus
GR = gyrus rectus
IRS = inferior rostral sulcus
LOS = lateral orbital sulcus
MOG = medial orbital gyrus
OFC = orbital frontal cortex
OFS = olfactory sulcus
ORS = orbital sulci
PS = pericallosal sulcus
SRS = superior rostral sulcus

Data courtesy of K. Botteron, Washington U. School of Medicine
VMPFC Coordinate System

Mean Curvature

GR
PS CG CS SRS IRS
Circle Packing Flexibility: Rectangular Discrete Conformal Maps

• An advantage of conformal mapping via circle packing is the flexibility to map a region to a desired shape.
• Boundary angles, rather than boundary radii are preserved.
• For a rectangle: 4 boundary vertices are nominated to act as the corners of the rectangle.
• Aspect ratio (width/height) is a conformal invariant of the surface (relative to the 4 corners) and is called the extremal length.
• Conformal extremal length represents one measure of shape.
• Two surfaces are conformally equivalent if and only if their conformal modulus is the same.
Euclidean Maps:
Specify Boundary Radius or Angle

Twin A
Left
Right

Twin B
Left
Right

3D Surface
Euclidean Map: Boundary Radius

Euclidean Map: Boundary Angle

Conformal Modulus
0.750
0.802

0.731
0.806
• Path of maximal curvature tracks along a gyrus / fold (top line).
• Path of minimal curvature tracks along a sulcus / fissure (bottom line).
Circle Packing Flexibility: Preserve Inversive Distance Rather than Circle Tangency

• Inversive distance between two oriented circles in the Riemann sphere is a conformal invariant of the location of the circles and their relative orientations

• As with tangency packings, inversive distance packings require radii of boundary circles or angle sums at boundary vertices to be specified
Inversive Distance

Let oriented circle $D$ be mutually orthogonal to oriented circles $C_1$ and $C_2$. Denote $z_1, z_2$ as the points of intersection of $D$ with $C_1$ and $w_1, w_2$ as the points of intersection of $D$ with $C_2$. The inversive distance between $C_1$ and $C_2$ is defined as:

- $\text{InvDist}(C_1, C_2) = 2[ z_1, z_2 ; w_1, w_2 ] - 1$
- $\text{InvDist}(C_1, C_2) = 1$ if $C_1$ and $C_2$ are tangent
- $\text{InvDist}(C_1, C_2) = \cos \alpha$, if $C_1$ and $C_2$ intersect with angle $\alpha$, $0 \leq \text{InvDist}(C_1, C_2) < 1$
- $\text{InvDist}(C_1, C_2) = \cosh \delta$, where $\delta$ is the hyperbolic distance between the hyperbolic planes bounded by disjoint circles $C_1$ and $C_2$, $1 < \text{InvDist}(C_1, C_2) < \infty$
Computing Inversive Distance
Circle Patterns (Rectangle Maps)

- $K = \text{triangulation of a disk with four distinguished boundary vertices with edge set } E \text{ and vertex set } V$;
- $\Phi: E \rightarrow [0, \infty)$ an inversive distance edge labeling

**Euclidean Formulation**

- For oriented circles $C_1$ and $C_2$ with radii $R_1$ and $R_2$ and centered at $a_1$ and $a_2$ respectively:
  \[
  \text{InvDist}(C_1, C_2) = \left( |a_1 - a_2|^2 - R_1^2 - R_2^2 \right) / 2R_1R_2
  \]
- Observe $|a_1 - a_2| = \text{edge length } e_{1,2} = \langle v_1, v_2 \rangle$
- For convergence, require $R_i$ to be a constant function. Thus:
  \[
  \text{InvDist}(C_1, C_2) = \Phi(e_{1,2}, R) = e^2/(2R^2) - 1
  \]

Existence questions??? Uniqueness proved by Luo, 2011
Example: Hexagonal Grid

- Lengths of bold edges are 1.1
- Other edge lengths are 1.4

- Now: adjust $R$ to construct a variety of overlapping, tangent, and disjoint circle packings using inversive distance
Disjoint Circles:

$R = 1/4$

$\Phi(e_{\text{bold}} = 1.1, R = 1/4) = 8.6800$

$\Phi(e_{\text{other}} = 1.4, R = 1/4) = 14.6800$
Overlapping Circles:

\[ R = \frac{1}{\sqrt{2}} \]

\[ \Phi(e_{\text{bold}} = 1.1, R = \frac{1}{\sqrt{2}}) = 0.2100 \]

\[ \Phi(e_{\text{other}} = 1.4, R = \frac{1}{\sqrt{2}}) = 0.9600 \]
Overlapping and Disjoint Circles: $R = \frac{3}{5}$

$\Phi(e_{\text{bold}} = 1.1, R = \frac{3}{5}) = 0.6806$

$\Phi(e_{\text{other}} = 1.4, R = \frac{3}{5}) = 1.7222$
Quadrilateral Subsurface: Inversive Distance Packings

Bump Map Texture
More Examples of Conformal Maps & Conformal Invariants
Conformal Maps in Neuroscience

Used in neuroimaging studies of

- Hippocampus
- Alzheimer’s disease
- Schizophrenia
- Cerebellum
- Cortical shape matching
- Hemispheric asymmetry

Used to model retinotopic mapping of visual cortex
Summary

• Circle packings are mathematically unique and converge to the discrete conformal map of a surface in the limit (i.e. through hex refinement) if triangulation is equilateral; otherwise yields an approximation to a discrete conformal map

• Euclidean, hyperbolic, spherical geometries available

• Flexible in terms of conformal mappings to shapes, circle tangency, inversive distance packings

• Inversive distance data allows more geometric information to be encoded

• Some known applications: brain mapping, tilings, Dessins

• Open questions remain regarding existence for inversive distance packings; theory proved for tangency & overlap packings with prescribed angles of overlap
Future Issues: Conformal Mapping in Neuroscience

• Compare maps between subjects: metrics
• Alignment of different regions
  – align one volume or surface in 3-space to another and then conformally flat map
  – conformally flat map 2 different surfaces and then align/morph one to the other (in 2D)
• Analysis of similarities, differences between different map regions
• Experiment with rectangular tangency versus inversive distance maps
• Other conformal invariants? Other applications?
Development of Cortical Folding Patterns Across Species

Linnaeus's Mouse Opossum
*Marmosa murina*

Domestic Cat
*Felis catus*

Ring-tailed Lemur
*Lemur catta*

Human
*Homo sapiens*

www.brainmuseum.org
Development of Sulcal Pattern

**Upper Layers:** Intermediate Progenitor Model (Kriegstein, 2006)
- only subsets of RGCs are activated to create IPCs
- result is non-uniform distribution of IPCs, which create local amplification of neuroblasts surrounded by areas of non-amplification

![Diagram of brain development]

R - Radial glial cell
N - Neuron
I - Intermediate progenitor cell

Application: Human Malformations: Polymicrogyria ("many small gyri")

- A neural migration disorder
- Characterized by an excessive number of small prominent convolutions spaced out by shallow and enlarged sulci.
- Many different types of polymicrogyria
  - Focal / Diffuse
  - Bilateral / Unilateral
  - Alone / Associated with other diseases
- "About 65% of patients [with polymicrogyria] have severe epilepsy" (Guerrini, 2006)
- Focal polymicrogyrias are often associated with malformations in GPR56 (regulates regional cortical patterning). (Rakic, 2004)

http://www.neuropathologyweb.org/chapter11/chapter11dNMD.html
Math Model: Turing Systems

- Modeling the development and growth of brain folding with a Turing reaction-diffusion system using dynamic growth
- Turing System: stability in the absence of diffusion and diffusion driven instability.

Reaction Diffusion System – BVM (Barrio, Varea, and Maini)

\[
\begin{align*}
\frac{du}{dt} &= \frac{D}{\rho^2} \nabla^2 u - 2Ru + \omega F(u, v) \\
\frac{dv}{dt} &= \frac{1}{\rho^2} \nabla^2 v - 2Rv + \omega G(u, v)
\end{align*}
\]

where

\[
F(u, v) = \alpha u(1 - r_3v^2) + v(1 - r_2u)
\]

\[
G(u, v) = \beta v \left(1 + \frac{\alpha r_3}{\beta} uv\right) + u(\gamma + r_2v)
\]

and \(u\) is an activator and \(v\) is an inhibitor

- \(d = \frac{D_u}{D_v} << 1\) ratio of diffusion terms \(\text{Constant}\)
- \(\omega = \text{domain scaling} \ Vary \omega\)
- \(\rho(t) = e^{Rt} = \text{growth rate function} \ Vary R\)
- \(\alpha, \beta = \text{linear interactions} \ Constant\)
- \(r_2, r_3 = \text{quadratic, cubic interactions} \ Constant\)

Pattern Formation

Modeling and understanding cortical folding pattern formation is important for quantifying cortical development.

“Healthy” Simulation

PMG Simulation
Modeling Cortical Diseases

NRS: $R=0.005$, $\omega=115$
Smaller ventricles

Normal: $R=0.015$, $\omega=115$

PMG: $R=0.021$, $\omega=115$
Enlarged ventricles
Our model is able to elucidate which parameters can lead to excessive cortical folding in disease.

Domain (focal distance) seems to play a role in sulcal pattern formation across species of increasing evolutionary complexity:
- Earlier in evolutionary timelines when LV focal distances are smaller, our model shows sectorial sulci appear before transverse sulci (e.g., domesticated cat, lemur, human).
- Later, when LV focal distances are larger, a first transverse sulcus appears, corresponding to calcarine sulcus (e.g., lemur, human).
- Later, when LV focal distances are even larger, a 2nd transverse sulcus appears, corresponding to central sulcus (e.g., human).

Modeling and understanding cortical folding pattern formation is important for quantifying cortical development.

Can applying conformal mapping to these simulated folding patterns contribute anything???
Acknowledgements

Collaborators
• De Witt Sumners, Phil Bowers (Math, FSU)
• Ken Stephenson (Math, UTK) & his CirclePack software

Data
• David Rottenberg (Radiology, Neurology, UMinnesota)
• Michael Miller, Center for Imaging Science (Biomed, JHU) & Kelly Botteron (Radiology, Psychiatry, WUSTL)

Funding
• NIH Human Brain Project Grant MH57180
• NSF Focused Research Group Grant 0101329
• Simons Foundation Collaboration Grant for Mathematicians

More Information on Brain Mapping:
• URL: http://www.math.fsu.edu/~mhurdal
• Email: mhurdal@math.fsu.edu