Multiresolution Analysis and Wavelets on Hierarchical Data Trees

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Introduction
Outline

- Introduction
- Construction of hierarchical data tree via data graph
Introduction

Construction of hierarchical data tree via data graph

Construction of wavelet basis and frame on hierarchical data
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Wavelet representations of functions on data set
Section 1. Introduction
The connection of sensor locations in US

Figure: Sensor locations inferred for $n = 1055$ largest cities in the continental US. On average, each sensor estimated local distances to 18 neighbors, with measurements corrupted by 10% Gaussian noise. We assume that the locations in the figure is not known in prior. Only the distance of two locations within radius of 0.1 can be measured.
The geometric structure of a data set is given by the weighted graph on the data.
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Let \( X \subset \mathbb{R}^D \) and \( |X| = n \). A weighted graph on \( X \) is the triple \( G = [X, E, W] \), where \( X \) is the node set, \( E \) is the edge set, and \( W \) is an \( n \times n \) (sparse) weight matrix with \( w_{i,j} = w_{j,i} \) and

\[
\begin{align*}
w_{i,j} & = 0, \quad (x_i, x_j) \notin E, \\
& > 0, \quad (x_i, x_j) \in E.
\end{align*}
\]
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- Let $X \subset \mathbb{R}^D$ and $|X| = n$. A weighted graph on $X$ is the triple $G = [X, E, W]$, where $X$ is the node set, $E$ is the edge set, and $W$ is an $n \times n$ (sparse) weight matrix with $w_{i,j} = w_{j,i}$ and
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  w_{i,j} > 0, & (x_i, x_j) \in E.
  \end{cases}
  \]
- Example: $w_{i,j} = \exp \left( -\frac{||x_i - x_j||^2}{2\sigma^2} \right)$, $(x_i, x_j) \in E$. 

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- Example: \( w_{i,j} = \exp \left( -\frac{\|x_i - x_j\|^2}{2\sigma^2} \right) \), \( (x_i, x_j) \in E \).

- The weight matrix defines a metric on the graph \( G \), which defines the kernel distance on \( X \):

\[
d^2_W(x_i, x_j) = w_{i,i} + w_{j,j} - 2w_{i,j}.
\]
On a connected data graph $G = [X, E, W]$, the weight is given by a positive definite and symmetric kernel $k(x_i, x_j) = w_{i,j}$. Let $d(x) = \int_X k(x, y) d\mu(y)$. 

Diffusion kernel: $\tilde{k}(x_i, y_j) = k(x_i, y_j) \cdot d(x_i) d(y_j)$. 

Compact support: $\sum_{i \leq n} \lambda_i = 1$ and $\lambda_1 \approx 1$. 

Diffusion map: It is defined as $\Phi_t(x) = \sum_{i \leq n} \lambda_i \phi_i(x) \Phi_t$. 

Diffusion distance: $d(\tilde{k})_t(x_i, y_j) = \Phi_t(x_i) \Phi_t(y_j)$. 

Wavelets on Data Trees
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  $$\Phi_t(x) = [\lambda_1^t \phi_1(x), \ldots, \lambda_{n-1}^t \phi_{n-1}(x)]^T.$$
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- **Diffusion map:** It is defined as $\{\Phi_t\} : X \to l^2$ such that
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- **Diffusion distance:** $d_{\tilde{k}}(x, y) = \|\Phi_t(x) - \Phi_t(y)\|$. 

Let $\mathcal{H} = L^2(X, \mu)$ be a Hilbert space of functions on $(X, \mu)$ and the diffusion operator on $\mathcal{H}$ be $(T^t f)(x) = \int_X \tilde{k}^t(x, y)f(y)d\mu(y)$. 

Let $\mathcal{H} = L^2(X, \mu)$ be a Hilbert space of functions on $(X, \mu)$ and the diffusion operator on $\mathcal{H}$ be $(T^tf)(x) = \int_X \tilde{k}_t(x, y)f(y)d\mu(y)$. Let $\epsilon > 0$ be sufficient small. A subspace $S \subset \mathcal{H}$ is called an $\epsilon$-null space of $T^t$ if $\|T^tf\| \leq \epsilon\|f\|$ for all $f \in S$. We denote it by $S = Nul_\epsilon(T^t)$. 
MRA on $\mathcal{H}$ and Diffusion Wavelets


Let $\mathcal{H} = L^2(X, \mu)$ be a Hilbert space of functions on $(X, \mu)$ and the diffusion operator on $\mathcal{H}$ be $(T^t f)(x) = \int_X \tilde{k}^t(x, y) f(y) d\mu(y)$.

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- Let $V_0 = \mathcal{H}$, $V_j = T^{2j-1} \mathcal{H}$, and $n \in \mathbb{N}$ be the integer such that $V_n = \text{Span}(v_0)$. Then

$$V_0 \supset V_1 \supset V_2 \supset \cdots \supset V_n$$

is an MRA of $\mathcal{H}$ with respect to $T$. 


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- Let $V_j = V_{j+1} \oplus W_{j+1}$, $V_{j+1} \perp W_{j+1}$. An o.n. basis of $W_j$ is call the diffusion wavelet basis of $W_j$. 

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Wavelets on Data Trees
MRA on $\mathcal{H}$ and Diffusion Wavelets


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- Let $V_0 = \mathcal{H}$, $V_j = T^{2^{j-1}}(\mathcal{H})$, and $n \in \mathbb{N}$ be the integer such that $V_n = \text{Span}(v_0)$. Then

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- The all basis of $W_j$ and $v_0$ form a basis of $\mathcal{H}$.
Ref. [D.K. Hammond, P. Vandergheynst, R. Gribonval, 2011]

Let $\mathcal{L}$ be the graph Laplacian on $G$ such that

$$\mathcal{L} = \sum_{j=0}^{n-1} \lambda_j \chi_j(x) \chi_j(y),$$

where $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_{n-1}$. 

The wavelet operator is defined as

$$T_g = g \sum_{j=0}^{n-1} \lambda_j \chi_j(x) \chi_j(y).$$

The spectral graph wavelet is defined as

$$\psi_t, x, y = \sum_{j=0}^{n-1} \lambda_j \chi_j(x) \chi_j(y),$$

where $t \in \mathbb{R}$. 

The wavelet transform of $f$ is given by

$$W_f = g \sum_{j=0}^{n-1} \lambda_j \chi_j(x) \chi_j(y).$$
Let $L$ be the graph Laplacian on $G$ such that

$$L = \sum_{j=0}^{n-1} \lambda_j \chi_j(x) \chi_j(y),$$

where $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_{n-1}$.

- Let $g$ be a function on $\mathbb{R}^+$. The wavelet operator is defined as

$$T_g = g(L) = \sum_{j=0}^{n-1} g(\lambda_j) \chi_j(x) \chi_j(y).$$
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- The spectral graph wavelet is defined as

$$\psi_{t,x}(y) = \sum_{j=0}^{n-1} g(t\lambda_j) \chi_j(x) \chi_j(y), \quad x \in X, \ t \geq 0.$$
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- The wavelet transform of $f$ is given by

$$W_f(t, x) = \langle \psi_{t,x}, f \rangle = \sum_{j=0}^{n-1} g(t\lambda_j) \chi_j(x) \sum_{y \in X} \chi_j(y) f(y).$$
Our Purpose

Constructing “traditional” compact supported wavelets on data set

- Construction of MRA on the data via hierarchical tree
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- Construction of MRA on the data via hierarchical tree
- Construction of compact supported wavelet basis and frame on data sets
Our Purpose

Constructing "traditional" compact supported wavelets on data set

- Construction of MRA on the data via hierarchical tree
- Construction of compact supported wavelet basis and frame on data sets
- Development of pyramid algorithm for wavelet decomposition and recovering of functions on data
Section 2. Construction of hierarchical data tree via data graph
We adopt the method proposed by [J. Shi and J. Malik, 2000]. Let $A, B, V \subset X$ s.t. $A \cap B = \emptyset$, $A \cup B = V$ and $A \subset V$. Define the cut of $(A, B)$ (w.r.t. $V$) and the association of $(A, V)$ as
\[
\text{cut}(A, B) = \sum_{a \in A, b \in B} k(a, b), \quad \text{assoc}(A, V) = \sum_{a \in A, v \in V} k(a, v)
\]
Data partition via normalized cut

We adopt the method proposed by [J. Shi and J. Malik, 2000]. Let $A, B, V \subset X$ s.t. $A \cap B = \emptyset, A \cup B = V$ and $A \subset V$. Define the cut of $(A, B)$ (w.r.t. $V$) and the association of $(A, V)$ as $\text{cut}(A, B) = \sum_{a \in A, b \in B} k(a, b)$, $\text{assoc}(A, V) = \sum_{a \in A, v \in V} k(a, v)$.

Definition

The normalized cut of $(A, B)$ (w.r.t. $V$) is the following number:

$$
\text{Ncut}(A, B) = \frac{\text{cut}(A, B)}{\text{assoc}(A, V)} + \frac{\text{cut}(A, B)}{\text{assoc}(B, V)}
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- $\text{Ncut}(A, B)$ can be naturally extended to $\text{Ncut}(A_1, \cdots, A_k)$. 
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\]

- \( Ncut(A, B) \) can be naturally extended to \( Ncut(A_1, \cdots, A_k) \).
- The optimal \( k \)-partition of \( V \) is the solution:

\[
(A_1, \cdots, A_k) = \arg \min Ncut(A_1, \cdots, A_k)
\]

where \( \bigcup_{j=1}^k A_j = V \) and \( A_i \cap A_j = \emptyset \), if \( i \neq j \).
The optimal $k$-partition is a NP problem, which can be relaxed to a problem of eigen-decomposition of $\mathcal{L}$ and approximatively solved as following:

1. Obtaining a $k$-dimensional reduction $Y \subseteq \mathbb{R}^k$ of $V$.
2. Making a $k$ partition of $Y$ using a clustering algorithm, say, the $k$-mean one.
3. Deriving the $k$-partition of $V$ from the $k$-partition of $Y$.

There are (self-tuning) algorithms for finding the optimal partition number $k$. [A. Zelnik-Manor and P. Perona, 2004.]

Applying the partition algorithm recursively, we construct a multi-layer partition, in which the cluster number $k$ can be varied for each subpartition.
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There are (self-tuning) algorithms for finding the optimal partition number $k$. [A. Zelnik-Manor and P. Perona, 2004.] Applying the partition algorithm recursively, we construct a multi-layer partition, in which the cluster number $k$ can be varied for each subpartition.
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Approximation of optimal partition

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Applying the partition algorithm recursively, we construct a multi-layer partition, in which the cluster number $k$ can be varied for each subpartition.
Construction of a hierarchical data tree

Definition

Assume $X$ has a $L$-layer partition s.t. $X = X^L_1 = \bigcup_{j=1}^{n_{L-1}} X^{L-1}_j$, and for $1 \leq \ell \leq L$, $X^\ell_k = \bigcup X^\ell_{j-1}$. Define $S^\ell = \{X^\ell_1, \cdots, X^\ell_{n_{\ell}}\}$, $1 \leq \ell \leq L$, and $S^0 = X$. Then the structure

$$S^L \prec S^{L-1} \prec \cdots \prec S^1 \prec S^0$$

is called a hierarchical data tree and denoted by $T(X)$. 

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Assume $X$ has a $L$-layer partition s.t. $X = X_1^L = \bigcup_{j=1}^{n_L-1} X_j^{L-1}$, and for $1 \leq \ell \leq L$, $X_k^\ell = \bigcup X_j^{\ell-1}$. Define

$S_\ell = \{X_1^\ell, \ldots, X_n^\ell\}$, $1 \leq \ell \leq L$, and $S_0 = X$. Then the structure

$$S_L \triangleleft S_{L-1} \triangleleft \cdots \triangleleft S_1 \triangleleft S_0$$

is called a hierarchical data tree and denoted by $\mathcal{T}(X)$.

- $S_L$ is called the *roof* of the tree $\mathcal{T}(x)$, the points in $X$ called the *leaves*, and a set in $S_\ell$ called a *$\ell$-level folder* (or a *$\ell$-level node*).
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Assume \( X \) has a \( L \)-layer partition s.t. \( X = X^L_1 = \bigcup_{j=1}^{n_{L-1}} X^{L-1}_j \), and for \( 1 \leq \ell \leq L \), \( X^\ell_k = \bigcup X^{\ell-1}_j \). Define

\[
{S_\ell} = \{X^\ell_1, \ldots, X^\ell_{n^\ell}\}, \quad 1 \leq \ell \leq L,
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is called a hierarchical data tree and denoted by \( \mathcal{T}(X) \).

- \( S_L \) is called the roof of the tree \( \mathcal{T}(x) \), the points in \( X \) called the leaves, and a set in \( S_\ell \) called a \( \ell \)-level folder (or a \( \ell \)-level node).
- The set \( X^k_j \) has a double identities: A subset of \( X \) and a \( k \)-level folder in the tree.
Construction of a hierarchical data tree

Definition

Assume $X$ has a $L$-layer partition s.t. $X = X_1^L = \bigcup_{j=1}^{n_L-1} X_j^{L-1}$, and for $1 \leq \ell \leq L$, $X_k^\ell = \bigcup X_j^{\ell-1}$. Define

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- $S_L$ is called the *roof* of the tree $\mathcal{T}(x)$, the points in $X$ called the *leaves*, and a set in $S_\ell$ called a $\ell$-level *folder* (or a $\ell$-level *node*).
- The set $X_j^k$ has a double identities: A subset of $X$ and a $k$-level folder in the tree.
- We have $\bigcup_k (X_k^\ell) = X$, $|S_0| = |X| = n$, $|S_L| = 1$. 

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Wavelets on Data Trees
By the partition tree, we can construct the hierarchical date tree. We apply an ordering operator to sort the nodes at each level, from the root to the leaves.

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- Balanced tree.
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For all parent and child folders,

\[ 0 < B \leq \frac{|\text{child folder}|}{|\text{parent folder}|} < \bar{B} < 1. \]
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  For all parent and child folders,
  
  \[ 0 < \mathcal{B} \leq \frac{|\text{child folder}|}{|\text{parent folder}|} \leq \overline{B} < 1. \]

For a balanced tree, the number of levels is \( L = \bigO(\log n) \).
Binary full data tree

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Full data tree II: Ternary tree

Ternary Full Tree
Tight Balanced Tree

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Wavelets on Data Trees
Let $A = \{a_1, a_2, \cdots, a_k\}$ and $B = \{b_1, b_2, \cdots, b_m\}$ be two folders at $(L - 1)$ level.

- Define the $k \times m$ distance matrix $D(A, B) = [d_G(a_i, b_j)]$. 

Let $A = \{a_1, a_2, \cdots, a_k\}$ and $B = \{b_1, b_2, \cdots, b_m\}$ be two folders at $(L-1)$ level.

- Define the $k \times m$ distance matrix $D(A, B) = [d_G(a_i, b_j)]$.
- The average distance $d_a(A, B) = \| D(A, B) \|_F$. 

Let $A = \{a_1, a_2, \cdots, a_k\}$ and $B = \{b_1, b_2, \cdots, b_m\}$ be two folders at $(L - 1)$ level.

- Define the $k \times m$ distance matrix $D(A, B) = [d_G(a_i, b_j)]$.
- The average distance $d_a(A, B) = \|D(A, B)\|_F$.
- The shortest distance $d_s(A, B) = \min d_G(a_i, b_j)$.
Distance between folders

Let $A = \{a_1, a_2, \cdots, a_k\}$ and $B = \{b_1, b_2, \cdots, b_m\}$ be two folders at $(L - 1)$ level.

- Define the $k \times m$ distance matrix $D(A, B) = [d_G(a_i, b_j)]$.
- The average distance $d_a(A, B) = \|D(A, B)\|_F$.
- The shortest distance $d_s(A, B) = \min d_G(a_i, b_j)$.
- The longest distance $d_l(A, B) = \max d_G(a_i, a_j)$.
Distance between folders

Let \( A = \{a_1, a_2, \cdots, a_k\} \) and \( B = \{b_1, b_2, \cdots, b_m\} \) be two folders at \((L - 1)\) level.

- Define the \( k \times m \) distance matrix \( D(A, B) = [d_G(a_i, b_j)] \).
- The average distance \( d_a(A, B) = \|D(A, B)\|_F \).
- The shortest distance \( d_s(A, B) = \min d_G(a_i, b_j) \).
- The longest distance \( d_l(A, B) = \max d_G(a_i, a_j) \).
Let $A = \{a_1, a_2, \cdots, a_k\}$ and $d$ a distance on $A$. Let $\pi$ be an index permutation of $[1, \cdots, k]$. We call $a_{\pi} = [a_{\pi(1)}, a_{\pi(2)}, \cdots, a_{\pi(k)}]$ a stack of $A$ headed by $a_{\pi(1)}$, and call $\ell(a_{\pi}) = \sum_{j=1}^{k-1} d(a_{\pi(j)}, a_{\pi(j+1)})$ the path length of $a_{\pi}$. We denote the set of permutations (with the head $l$) by

$$P_l = \{\pi; \; \pi(1) = l\}.$$ 

**Definition**

A shortest-path sorting of $A$ headed by $a_l$ is a stack $a_{\pi}$ that has the shortest path length among all pathes starting from $a_l$:

$$a_{\pi} = \arg\min_{\pi \in P_l} \ell(a_{\pi}).$$
Denote by $A$ the folder set at a level. Let $p$ be a probability function on $A$ and $\Omega$ the sorted index set initialized to $\Omega = \emptyset$.

1. Set $\pi(1) = l$ and update $\Omega = \{l\}$. 
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1. Set $\pi(1) = l$ and update $\Omega = \{l\}$.

2. After $i$ steps, assume now $\Omega = \{\pi(1), \cdots, \pi(i)\}$. To find $\pi(i+1)$, from unsorted elements, pick up two nearest ones $y_1$ and $y_2$ of $a_{\pi(i)}$ and compute

$$q_i = \frac{1}{1 + \exp \left( \frac{d(a_i,y_1) - d(a_i,y_2)}{\alpha} \right)},$$

where $\alpha > 0$ is the sorting parameter. If $q_i < p_{\pi(i)}$, we select $a_{\pi(i+1)} = y_2$. Otherwise, select $a_{\pi(i+1)} = y_1$. 

3. Update $\Omega$, and repeat the step above. The algorithm is terminated when $|\Omega| = k$.

4. $\pi$ is an approximative shortest path sorting of $A$ headed by $l$. 

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*Wavelets on Data Trees*
Denote by $A$ the folder set at a level. Let $p$ be a probability function on $A$ and $\Omega$ the sorted index set initialized to $\Omega = \emptyset$.

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where $\alpha > 0$ is the sorting parameter. If $q_i < p_{\pi(i)}$, we select $a_{\pi(i+1)} = y_2$. Otherwise, select $a_{\pi(i+1)} = y_1$.

3. Update $\Omega$, and repeat the step above. The algorithm is terminated when $|\Omega| = k$.

4. $a_{\pi}$ is an approximative shortest path sorting of $A$ headed by $a_l$. 
Algorithm for building data tree

1. Input: A weighted graph \( G = [X, W] \) on the data set \( X \).
2. Construct the matrix \( P = D^{-1}W \) and use a fast eigen-decomposition algorithm to find the largest \( k \) Left eigenvectors. To make sure that the gap between \( \lambda_k \) and \( \lambda_{k+1} \) is large.
3. Use a partition algorithm, e.g., \( k \)-mean, to make a partition of \( X = \{x_1, \cdots, x_n\} \).
4. On each subset \( X_j \), repeat the processing above to partition it again up to \( L \) levels.
5. Smoothly order the folders at each level.
Data tree of a brain image

Figure: Data tree of a brain image.
Section 3. Construction of hierarchical data tree via data graph
Definition

Let $\mathcal{H}_0 = \mathcal{H}(= L^2(X, d\mu))$ and $\mathcal{H}_\ell = \{f \in \mathcal{H}; f(x) = c_j, x \in X^\ell_j \in S_\ell\}$. The hierarchical tree $T(X)$ derives the following MRA on $\mathcal{H}$:

$$\mathcal{H}_0 \supset \mathcal{H}_1 \cdots \supset \mathcal{H}_L$$

where $\dim(\mathcal{H}_\ell) = n_\ell (= |S_\ell|)$. 

MRA on $\mathcal{H}$ via a hierarchical data tree
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**Definition**

Let $\mathcal{H}_0 = \mathcal{H}(= L^2(X, d\mu))$ and $\mathcal{H}_\ell = \{ f \in \mathcal{H}; f(x) = c_j, x \in X_j^\ell \in S_\ell \}$. The hierarchical tree $T(X)$ derives the following MRA on $\mathcal{H}$:

$$\mathcal{H}_0 \supset \mathcal{H}_1 \cdots \supset \mathcal{H}_L$$

where $\dim(\mathcal{H}_\ell) = n_\ell (= |S_\ell|)$. Let $\mathcal{W}_\ell \oplus \mathcal{H}_\ell = \mathcal{H}_{\ell-1}$ and $\mathcal{W}_\ell \perp \mathcal{H}_\ell$. Then $\mathcal{W}_\ell$ is a wavelet subspace of $\mathcal{H}$. 
**Definition**

Let \( \mathcal{H}_0 = \mathcal{H}(= L^2(X, d\mu)) \) and 
\( \mathcal{H}_\ell = \{ f \in \mathcal{H}; \ f(x) = c_j, x \in X_j^\ell \in S_\ell \} \). The hierarchical tree \( \mathcal{T}(X) \) derives the following MRA on \( \mathcal{H} \):

\[
\mathcal{H}_0 \supset \mathcal{H}_1 \cdots \supset \mathcal{H}_L
\]

where \( \dim(\mathcal{H}_\ell) = n_\ell (= |S_\ell|) \). Let \( \mathcal{W}_\ell \oplus \mathcal{H}_\ell = \mathcal{H}_{\ell-1} \) and \( \mathcal{W}_\ell \perp \mathcal{H}_\ell \). Then \( \mathcal{W}_\ell \) is a wavelet subspace of \( \mathcal{H} \).

We have \( \dim(\mathcal{W}_\ell) = m_\ell = |S_{\ell-1}| - |S_\ell| \), and

\[
\mathcal{H} = \mathcal{H}_L \oplus \mathcal{W}_L \oplus \cdots \oplus \mathcal{W}_1.
\]
In $L^2(X, dx)$, $\langle a, b \rangle = \sum_j a_j b_j$.

In $L^2(X, d\mu)$, $\langle a, b \rangle_m = \sum_j a_j b_j m_j = \langle a, bm \rangle$. 
The relation between o.n. bases of $L^2(X, d\mu)$ and of $L^2(X, dx)$

1. In $L^2(X, dx)$, $\langle a, b \rangle = \sum_j a_j b_j$.
   In $L^2(X, d\mu)$, $\langle a, b \rangle_m = \sum_j a_j b_j m_j = \langle a, b_m \rangle$.

2. Let $\{\eta_j\}_{j=1}^n$ be an o.n. basis of $L^2(X, dx)$. Then, setting $\tilde{\eta}_j = \eta_j / \sqrt{m}$, $\{\tilde{\eta}_j\}_{j=1}^n$ is an o.n. basis of $L^2(X, d\mu)$.
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   $\tilde{\eta}_j = \eta_j / \sqrt{m}$, $\{\tilde{\eta}_j\}_{j=1}^n$ is an o.n. basis of $L^2(X, d\mu)$.

3. We may use o.n wavelet basis of $L^2(X, dx)$ to perform the o.n. wavelet decomposition and recovering for $f \in L^2(X, d\mu)$ by using the following formula:
   $$\langle fm, \eta_j \rangle = \langle f, \tilde{\eta}_j \rangle_m.$$
The scaling functions and wavelet functions in $H = L^2(X, dx)$ have the following properties:

**Properties of scaling function and wavelets**

- At the leaf level, the set of delta functions $\{\delta_x\}_{x \in X}$ is an o.n. basis of $H$. Each $f \in H$ has the decomposition $f = \sum_j f_j^0 \delta_{x_j}$, where $f_j^0 = f(x_j)$. 

Hierarchical structure of wavelet basis on $\mathcal{H} = L^2(X, dx)$

The scaling functions and wavelet functions in $\mathcal{H} = L^2(X, dx)$ have the following properties:

**Properties of scaling function and wavelets**

- At the leaf level, the set of delta functions $\{\delta_x\}_{x \in X}$ is an o.n. basis of $\mathcal{H}$. Each $f \in \mathcal{H}$ has the decomposition $f = \sum_j f_j^0 \delta_{x_j}$, where $f_j^0 = f(x_j)$.
- At Level $\ell$, assume $\mathcal{S}_\ell = \{X_j^\ell\}_{j=1}^{n_\ell}$. Let

$$\phi_j^\ell(x) = \begin{cases} \frac{1}{\sqrt{|X_j^\ell|}}, & x \in X_j^\ell, \\ 0, & x \notin X_j^\ell. \end{cases}$$

Then $\{\phi_j^\ell\}_{j=1}^{n_\ell}$ is an o.n. basis of $\mathcal{H}_\ell$. 
Hierarchical structure of wavelet basis on $\mathcal{H} = L^2(X, dx)$

The scaling functions and wavelet functions in $\mathcal{H} = L^2(X, dx)$ have the following properties:

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- At the leaf level, the set of delta functions $\{\delta_x\}_{x \in \mathcal{X}}$ is an o.n. basis of $\mathcal{H}$. Each $f \in \mathcal{H}$ has the decomposition $f = \sum_j f^0_j \delta_{x_j}$, where $f^0_j = f(x_j)$.

- At Level $\ell$, assume $S_\ell = \{X^\ell_j\}_{j=1}^{n_\ell}$. Let
  \[
  \phi^\ell_j(x) = \begin{cases} 
  \frac{1}{\sqrt{|X^\ell_j|}}, & x \in X^\ell_j, \\
  0, & x \notin X^\ell_j.
  \end{cases}
  \]
  Then $\{\phi^\ell_j\}_{j=1}^{n_\ell}$ is an o.n. basis of $\mathcal{H}_\ell$.

- There is a wavelet basis $\{\psi^\ell_j\}_{j=1}^{m_\ell}$ of $\mathcal{W}_\ell$ such that each $\psi^\ell_j$ is locally supported and has at least one vanishing moment, i.e., there is $1 \leq s \leq m_\ell$, s.t. $\text{supp}(\psi^\ell_j) \subset X^\ell_s$, and $\langle \psi^\ell_j, 1 \rangle = 0$. 

Jianzhong Wang

Wavelets on Data Trees
By the properties of wavelets, we may construct the wavelet basis on $\mathcal{H}$ folder-by-folder. We denote by $Y$ a folder at 1-level having $k$ leaves: $Y = \{y_j\}_{j=1}^k$. Let $\phi_j^0 = \delta_{y_j}$. Then $\{\phi_j^0\}_{j=1}^k$ is an o.n. basis of $L^2(Y, dy)$. The spatial representation of $f \in L^2(Y, dy)$ is $f = \sum_{j=1}^k f_j \phi_j^0$. We denote by $f$ the vector $[f_1, \cdots, f_k]^T$ too.
By the properties of wavelets, we may construct the wavelet basis on $\mathcal{H}$ folder-by-folder. We denote by $Y$ a folder at 1-level having $k$ leaves: $Y = \{y_j\}_{j=1}^k$. Let $\phi_j^0 = \delta_{y_j}$. Then $\{\phi_j^0\}_{j=1}^k$ is an o.n. basis of $L^2(Y, dy)$. The spatial representation of $f \in L^2(Y, dy)$ is $f = \sum_{j=1}^k f_j \phi_j^0$. We denote by $f$ the vector $[f_1, \cdots, f_k]^T$ too.

**Definition**

An o.n. wavelet basis on $L^2(Y, dy)$ is a $k \times k$ o.g. matrix: $M = [\phi, \psi_1, \cdots, \psi_{k-1}]$, where the first column $\phi$ is a scaling function and others are wavelets. The wavelet transform of a function $f \in L^2(Y, dy)$ is given by $d = M^T f$ and the inverse wavelet transform given by in $f = Md$.

By MRA on $L^2(Y)$, we may construct the o.n. wavelet basis of $L^2(Y)$ using a pyramid algorithm.
Let the first layer Haar o.n. wavelet basis be represented as a $k \times k$ matrix $M_1 = [L_1, H_1]$, where $L_1 = [\phi_1^1, \cdots, \phi_{k/2}^1]$ contains scaling functions and $H = [\psi_1^1, \cdots, \psi_{(k+1)/2}^1]$ contains wavelets.
Let the first layer Haar o.n. wavelet basis be represented as a $k \times k$ matrix $M_1 = [L_1, H_1]$, where $L_1 = [\phi_1^1, \cdots, \phi_{[k/2]}^1]$ contains scaling functions and $H = [\psi_1^1, \cdots, \psi_{[k+1]/2}^1]$ contains wavelets.

**Construction I: From 2 leaf scaling functions**

$$[\phi_i^1, \psi_i^1] = [\phi_{2i-1}^0, \phi_{2i}^0] \begin{bmatrix} \frac{1}{\sqrt{2}} & 1 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$
Let the first layer Haar o.n. wavelet basis be represented as a $k \times k$ matrix $M_1 = [L_1, H_1]$, where $L_1 = [\phi_1^1, \cdots, \phi_{[k/2]}^1]$ contains scaling functions and $H = [\psi_1^1, \cdots, \psi_{[k+1]/2}^1]$ contains wavelets.

**Construction I:** From 2 leaf scaling functions

\[
[\phi_i^1, \psi_i^1] = [\phi_{2i-1}^0, \phi_{2i}^0] \begin{bmatrix}
\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\
\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}
\end{bmatrix}
\]

When $k = 2s - 1$, we also need the following:

**Construction II:** From 3 leaf scaling functions

\[
[\phi_{s-1}^1, \psi_{s-1}^1, \psi_s^1] = [\phi_{m-2}^0, \phi_{m-1}^0, \phi_m] \begin{bmatrix}
\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\
\frac{1}{\sqrt{3}} & 0 & -\frac{2}{\sqrt{6}} \\
\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}}
\end{bmatrix}.
\]
We now construct \((j + 1)\)-level scaling functions and wavelets from \(j\)-level scaling functions \(\Phi_j = [\phi_1^j, \cdots, \phi_m^j]\). Write \(s_i = |\text{supp}(\phi_i^j)|\).

\[
\begin{bmatrix}
\phi_i^{j+1} \\
\psi_i^{j+1}
\end{bmatrix} = \begin{bmatrix}
\phi_{2i-1}^j \\
\phi_{2i}^j
\end{bmatrix} W_j^2
\]

\[
W_j^2 = \frac{1}{\sqrt{s_{2i-1}^2 + s_{2i+2}^2}} \begin{bmatrix}
\sqrt{s_{2i-1}} & \sqrt{s_{2i}} \\
\sqrt{s_{2i}} & -\sqrt{s_{2i-1}}
\end{bmatrix}
\]
We now construct \( (j+1) \)-level scaling functions and wavelets from \( j \)-level scaling functions \( \Phi_j = [\phi_1^j, \cdots, \phi_m^j] \). Write \( s_i = |\text{supp}(\phi_i^j)| \).

\[
[\phi_i^{j+1}, \psi_i^{j+1}] = [\phi_{2i-1}^j, \phi_{2i}^j] W_j^2
\]

\[
W_j^2 = \frac{1}{\sqrt{s_{2i-1}+s_{2i}}} \begin{bmatrix}
\sqrt{s_{2i-1}} & \sqrt{s_{2i}} \\
\sqrt{s_{2i}} & -\sqrt{s_{2i-1}}
\end{bmatrix}
\]

Wavelet transform: \( c^{j+1} = (W_j^2)^T c^j \), \( c^j = W_j^2 c^{j+1} \).
We now construct $(j + 1)$-level scaling functions and wavelets from $j$-level scaling functions $\Phi_j = [\phi_1^j, \cdots, \phi_m^j]$. Write $s_i = |\text{supp}(\phi_i^j)|$.

$$[\phi_{i+1}^j, \psi_{i+1}^j] = [\phi_{2i-1}^j, \phi_{2i}^j] W_j^2$$

$$W_j^2 = \frac{1}{\sqrt{s_{2i-1} + s_{2i}}} \begin{bmatrix} \sqrt{s_{2i-1}} & \sqrt{s_{2i}} \\ \sqrt{s_{2i}} & -\sqrt{s_{2i-1}} \end{bmatrix}$$

Wavelet transform: $c^{j+1} = (W_j^2)^T c^j, \ c^j = W_j^2 c^{j+1}$.

When $m = 2s - 1$, set $h_m = s_{m-2} + s_{m-1} + s_m$.

$$[\phi_{s-1}^{j+1}, \psi_{s-1}^{j+1}, \psi_{s}^{j+1}] = [\phi_{m-2}^j, \phi_{m-1}^j, \phi_m^j] W_j^3$$

$$W_j^3 = \begin{bmatrix} \sqrt{s_{m-2}/h_m} & \sqrt{s_m/s_{m-2}+s_m} & \sqrt{s_{m-1}s_m/s_{m-2}+s_m} \\ \sqrt{s_{m-1}/h_m} & 0 & -\sqrt{s_{m-2}+s_m/h_m} \\ \sqrt{s_m/h_m} & -\sqrt{s_{m-2}/s_{m-2}+s_m} & \sqrt{s_{m-1}s_m/s_{m-2}+s_m} \end{bmatrix}.$$
We now construct \((j + 1)\)-level scaling functions and wavelets from \(j\)-level scaling functions \(\Phi_j = [\phi^j_1, \ldots, \phi^j_m]\). Write \(s_i = |\text{supp}(\phi^j_i)|\).

\[
[\phi_{i}^{j+1}, \psi_{i}^{j+1}] = [\phi_{2i-1}^{j}, \phi_{2i}^{j}] W_j^2
\]

\[
W_j^2 = \frac{1}{\sqrt{s_{2i-1} + s_{2i}}} \begin{bmatrix}
\sqrt{s_{2i-1}} & \sqrt{s_{2i}} \\
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When \(m = 2s - 1\), set \(h_m = s_{m-2} + s_{m-1} + s_m\).

\[
[\phi_{s-1}^{j+1}, \psi_{s-1}^{j+1}, \psi_s^{j+1}] = [\phi_{m-2}^{j}, \phi_{m-1}^{j}, \phi_m^{j}] W_j^3
\]

\[
W_j^3 = \begin{bmatrix}
\sqrt{\frac{s_{m-2}}{h_m}} & \sqrt{\frac{s_m}{s_{m-2} + s_m}} & \sqrt{\frac{s_{m-1}s_{m-2}}{h_m(s_{m-2} + s_m)}} \\
\sqrt{\frac{s_{m-1}}{h_m}} & 0 & -\sqrt{\frac{s_{m-2} + s_m}{h_m}} \\
\sqrt{\frac{s_m}{h_m}} & -\sqrt{\frac{s_{m-2}}{s_{m-2} + s_m}} & \sqrt{\frac{s_{m-1}s_m}{h_m(s_{m-2} + s_m)}}
\end{bmatrix}
\]

Wavelet transform: \(c^{j+1} = (W_j^3)^T c^j\), \(c^j = W_j^3 c^{j+1}\).
The construction of wavelets above can be applied to the whole tree. Assume that the Haar wavelet basis has been built up to Level $\ell$, where $S_\ell = \{X_1^\ell, \cdots, X_{n_\ell}^\ell\}$. Therefore, in this basis, there are $n_\ell$ scaling functions: \[
\phi_j^{(\ell)} = \frac{1}{\sqrt{|X_j^\ell|}} \chi_{X_j^\ell}, \quad 1 \leq j \leq n_\ell.
\]
Let a wavelet on $X_k^\ell$ is denoted by $\psi_j^{(\ell,k)}$. (If it is at $i$-th layer and the layer level need to stress, then it is denoted by $\psi_{i,j}^{(\ell,k)}$.)
The construction of wavelets above can be applied to the whole tree. Assume that the Haar wavelet basis has been built up to Level \( \ell \), where \( S_\ell = \{X_1^\ell, \cdots, X_{n_\ell}^\ell\} \). Therefore, in this basis, there are \( n_\ell \) scaling functions: \( \phi_j^{(\ell)} = \frac{1}{\sqrt{|X_j^\ell|}} \chi_{X_j^\ell}, 1 \leq j \leq n_\ell \). Let a wavelet on \( X_k^\ell \) is denoted by \( \psi_j^{(\ell,k)} \). (If it is at \( i \)-th layer and the layer level need to stress, then it is denoted by \( \psi_{i,j}^{(\ell,k)} \).)

Let \( X_1^{\ell+1} = \bigcup_{j=1}^k X_j^\ell, X_1^{\ell+1} \in S_{\ell+1} \). We construct the \((\ell+1)\)-layer wavelets on \( X_1^{\ell+1} \) recursively.

- Initialize 0-layer wavelets as \( \phi_{0,j}^{(\ell+1,1)} = \phi_j^{(\ell)}, 1 \leq j \leq k \).
The construction of wavelets above can be applied to the whole tree. Assume that the Haar wavelet basis has been built up to Level $\ell$, where $S_\ell = \{X_1^\ell, \cdots, X_{n_\ell}^\ell\}$. Therefore, in this basis, there are $n_\ell$ scaling functions: $\phi_j^{(\ell)} = \frac{1}{\sqrt{|X_j^\ell|}} \chi_{X_j^\ell}$, $1 \leq j \leq n_\ell$. Let a wavelet on $X_k^\ell$ is denoted by $\psi_j^{(\ell,k)}$. (If it is at $i$-th layer and the layer level need to stress, then it is denoted by $\psi_j^{(\ell,k,i)}$.) Let $X_1^{\ell+1} = \bigcup_{j=1}^k X_j^\ell$, $X_1^{\ell+1} \in S_{\ell+1}$. We construct the $(\ell+1)$-layer wavelets on $X_1^{\ell+1}$ recursively.

- Initialize 0-layer wavelets as $\phi_{0,j}^{(\ell+1,1)} = \phi_j^{(\ell)}$, $1 \leq j \leq k$.
- When $k$ is even, then apply

$$[\phi_{t+1,i}^{(\ell+1,1)}, \psi_{t+1,i}^{(\ell+1,1)}] = [\phi_{t,2i-1}^{(\ell+1,1)}, \phi_{t,2i}^{(\ell+1,1)}] W_j^2$$
The construction of wavelets above can be applied to the whole tree. Assume that the Haar wavelet basis has been built up to Level $\ell$, where $S_\ell = \{X_1^\ell, \ldots, X_n^\ell\}$. Therefore, in this basis, there are $n_\ell$ scaling functions: $\phi_j^{(\ell)} = \frac{1}{\sqrt{|X_j^\ell|}} \chi_{X_j^\ell}, 1 \leq j \leq n_\ell$. Let a wavelet on $X_k^\ell$ is denoted by $\psi_j^{(\ell,k)}$. (If it is at $i$-th layer and the layer level need to stress, then it is denoted by $\psi_{i,j}^{(\ell,k)}$.)

Let $X_1^{\ell+1} = \bigcup_{j=1}^k X_j^\ell, X_1^{\ell+1} \in S_{\ell+1}$. We construct the $(\ell + 1)$-layer wavelets on $X_1^{\ell+1}$ recursively.

- Initialize 0-layer wavelets as $\phi_{0,j}^{(\ell+1,1)} = \phi_j^{(\ell)}$, $1 \leq j \leq k$.
- When $k$ is even, then apply
  \[
  \begin{bmatrix} \phi_t^{(\ell+1,1)} & \psi_t^{(\ell+1,1)} \end{bmatrix} = \begin{bmatrix} \phi_t^{(\ell+1,1)} & \phi_t^{(\ell+1,1)} \end{bmatrix} W_j^2
  \]
- When $k = 2s - 1$, we apply following for the last block:
  \[
  \begin{bmatrix} \phi_t^{(\ell+1,1)} & \psi_t^{(\ell+1,1)} & \psi_t^{(\ell+1,1)} \end{bmatrix} = \begin{bmatrix} \phi_t^{(\ell+1,1)} & \phi_t^{(\ell+1,1)} & \phi_t^{(\ell+1,1)} \end{bmatrix} W_j^3
  \]
Using the similar way, we also can construct a tight frame on the data tree $\mathcal{T}(X)$.

**Motivation**
- Tight frames have excellent localization.
- The redundance in the frames are very useful in data analysis and processing.
- Rich algorithms and methods for constructions of tight frames with boundaries are available in literature. Ref. [Chan, Riemenschneider, Shen, and Shen, 1998; Cai, Chan, Shen, and Shen, 1998; Daubechies, Han, Ron and Shen, 2003; Shen, 2010; ...].

**The steps for constructing tight frame on a data tree**
- Construction of tight frame within a folder.
- Construction of tight frame on the whole tree.
To construct the wavelet tight frame within a folder, we employ the tight framelets on a space of finite sequence $[x_1, \cdots, x_N]$ with a certain boundary condition, say, symmetric one. [see Chan, Riemenschneider, Shen, and Shen, 2005]
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- When \(L = 3\), choose \(h_0 = [1/4, 1/2, 1/4]\), \(h_1 = [-1/4, 1/2, -1/4]\), \(h_2 = [-\sqrt{2}/4, 0, \sqrt{2}/4]\) as the masks of the generators for the tight frame \([\phi, \psi_1, \psi_2]\).
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- When \(L = 4\), choose \(h_0 = \frac{1}{8}[1, 2, 2, 2, 1]\), \(h_1 = \frac{1}{8}[1, 0, 0, 0, -1]\), \(h_2 = \frac{\sqrt{2}}{8} \cos \left(\frac{\pi}{8}\right) [1, \sqrt{2}, 0, -\sqrt{2}, -1]\), \(h_3 = \frac{\sqrt{2}}{8} [\cos \left(\frac{\pi}{8}\right), -\sqrt{2} \sin \left(\frac{\pi}{8}\right), -2 \sin \left(\frac{\pi}{8}\right), -\sqrt{2} \sin \left(\frac{\pi}{8}\right), \cos \left(\frac{\pi}{8}\right)]\) \(h_4 = \frac{1}{8}[1, 0, -2, 0, 1]\), \(h_5 = \frac{1}{8}[1, -2, 0, 2, -1]\), \(h_6 = \frac{\sqrt{2}}{8} \sin \left(\frac{\pi}{8}\right) [1, -\sqrt{2}, 0, \sqrt{2}, -1]\), \(h_7 = \frac{\sqrt{2}}{8} [\sin \left(\frac{\pi}{8}\right), -\sqrt{2} \cos \left(\frac{\pi}{8}\right), -2 \cos \left(\frac{\pi}{8}\right), -\sqrt{2} \cos \left(\frac{\pi}{8}\right), \sin \left(\frac{\pi}{8}\right)]\)
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- The boundary elements need to add.
At a level $\ell$, Assume the the coefficient sequence of scaling functions is $c = [c_1, \cdots, c_N], N \geq 5$. When $N$ is odd, we choose the framelets with $L = 3$ and when it is even, we choose them with $L = 4$.

- If $1 < N < 5$, then we use the Haar do construct the wavelet and scaling function.
- The result tight frame within the folder contains only one scaling function.
To decompose the data in a tree by tight frame, we introduce the following:

**Definition**

Let $\mathcal{T}(X)$ be a data tree on the space $(X, d\mu_0)$, where $d\mu_0 = m^{(0)} dx$ and $m^{(0)}$ is a measure function. Assume also $\mathcal{T}(X)$ has $L$ levels: $S_L \triangleright S_{L-1} \triangleright \cdots \triangleright S_1 \triangleright S_0$. Then the measure function $m^{(\ell)}$ on $(S_\ell, d\mu_\ell)$ is defined as

$$m^{(\ell)}(X_\ell^k) = \sum_{X_{j}^{\ell-1} \subseteq X_{k}^{\ell}} m^{\ell-1}(X_{j}^{\ell-1}),$$

and the set $\{m^{(0)}, \cdots, m^{(L)}\}$ is called a hierarchical measures on the tree $\mathcal{T}(X)$. 

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Wavelets on Data Trees
Example

Let $m^{(0)}$ be the uniform measure such that $m^{(0)}(x) = 1, x \in X$. Then $m^{(\ell)}(X_j^\ell) = |X_j^\ell|$. It can be normalized to pmf by setting

$$p^{(\ell)}(X_j^\ell) = \frac{|X_j^\ell|}{|X|}.$$
1. Within each folder, construct the tight frame as described above.

2. For cross-level folders, we make the tight frame w.r.t. the measure $m$. Let $\{\eta_j\}_{j=1}^n$ be an tight frame of $L^2(X, dx)$. Write $\tilde{\eta}_j = \eta_j / \sqrt{m}$. Then $\{\tilde{\eta}_j\}_{j=1}^n$ is an tight of $L^2(X, d\mu)$. 
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3. To compute the coefficients of the tight frame on $L^2(X, d\mu)$, we use the formula:

$$\langle fm, \eta_j \rangle = \langle f, \tilde{\eta}_j \rangle_m.$$
Section 4. Wavelet representations of functions on data set
Why do we need wavelet representation?

1. It works on a wide-range of data sets and avoids to treat the high-dimensional data directly. (No curse of dimensionality).
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1. It works on a wide-range of data sets and avoids to treat the high-dimensional data directly. (No curse of dimensionality).
2. It needn’t the spatial operators that work only on the data sets in $\mathbb{R}^D$.
3. It provides sparse representations of the functions such as compactly supported functions, piecewise constant functions, zero-moment functions, and so on.
4. The optimization models based on wavelets usually have simple structure and lead to a fast algorithm.
Compute the wavelet coefficients via pyramid algorithm.

Let the data tree on $X$ be given:

$$X = X^L_1 \supseteq \{X^{L-1}_1, \ldots, X^{L-1}_{n_1}\} \supseteq \cdots \supseteq \{X^0_1, \ldots, X^0_n\},$$

where $X^0_j = \{x_j\}$. Assume that the wavelet o.n. basis or the tight wavelet frame is constructed. Let $f \in L^2(X)$. We may apply the classical Mallat's pyramid algorithm to compute the wavelet coefficients of $f$.

- As the initial, we set $c = [c_1, \cdots, c_n] = [f(x_1), \cdots, f(x_n)]$.

Then $f = \sum_{j=1}^{n} c_j \phi_j^0(x)$, where $\phi_j^0(x_i) = \delta_{i,j}$. 
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- At Level 1, assume that $X^1_1 = \{x_1, \cdots, x_m\}$ and the Haar o.n. basis is employed. Denote by $c_1 = [c_1, \cdots, c_m]$. Then

  $c_{1,1} = (\downarrow 2)c_1 * h_0$, $d_{1,1} = (\downarrow 2)c_1 * h_1$ and

  $c_{1,2} = (\downarrow 2)c_{1,1} * h_0$, $d_{1,2} = (\downarrow 2)c_{1,1} * h_1$
Compute the wavelet coefficients via pyramid algorithm

Let the data tree on $X$ be given:

$$X = X_1^L \supseteq \{X_1^{L-1}, \ldots, X_m^{L-1}\} \supseteq \cdots \supseteq \{X_1^0, \ldots, X_n^0\},$$

where $X_j^0 = \{x_j\}$. Assume that the wavelet o.n. basis or the tight wavelet frame is constructed. Let $f \in L^2(X)$. We may apply the classical Mallat’s pyramid algorithm to compute the wavelet coefficients of $f$.

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- The decompositions are repeated, say $K_1$ times, until $c_{1,K_1}$ is reduced to a single value.
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- The decompositions are repeated, say $K_1$ times, until $c_{1,K_1}$ is reduced to a single value.
- Repeat the steps above for $[c_1,K_1, \cdots, c_{n_{L-1}},K_{n_{L-1}}]$ and so on.
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- The reconstruction of $f$ from its wavelet coefficients is also similar to the classical pyramid algorithm.
The similar algorithm is available for tight wavelet frame too. The reconstruction of \( f \) from its wavelet coefficients is also similar to the classical pyramid algorithm. In the wavelet representation \( f = c_L \phi^L + \sum d_{\ell,k,j} \psi_{\ell,k,j}, \) \( c_0 \) is the average of \( f: \) \( c_0 = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} f(x_j). \) We denote by \( W_f \) for the vector of wavelet coefficients of \( f \) (excluding \( c_L \)).
Ref. [M. Gavish, B. Nadler, R.R. Coifman, 2010]

Definition

For each subset \( S \subset X \), define \( \rho(S) = |S|/|X| \). For \( x, y \in X \), denote by \( S(x, y) \) the smallest folder in the tree \( T(X) \) that contains both \( x \) and \( y \). Then the tree distance of \( x \) and \( y \) is defined as

\[
d_T(x, y) = \begin{cases} 
\rho(S(x, y)), & x \neq y, \\
0, & x = y. 
\end{cases}
\]

For \( 0 < \alpha < 1 \), a function \( f \in L^2(X) \) is called \( \alpha \)-Hölder continuous w.r.t. \( T \) (denoted by \( f \in H^\alpha(T) \)) if

\[
|f(x) - f(y)| \leq C d_T^\alpha(x, y), \quad \forall x, y, \in X.
\]
Theorem

Assume \( f \in \mathcal{H}^\alpha(T) \) and \( \psi_j^{(\ell,k)} \) is the wavelet at \( \ell \)-level with \( \text{supp}(\psi_j^{(\ell,k)}) \subset X_k^\ell \). Then

\[
\langle f, \psi_j^{(\ell,k)} \rangle \leq C \rho(X_k^\ell)^{\alpha+1/2}.
\]

On the other hand, if the inequality above holds for all wavelets \( \psi_j^{(\ell,k)} \), then \( f \in \mathcal{H}^\alpha(T) \).
Theorem

Assume $f \in \mathcal{H}^{\alpha}(\mathcal{T})$ and $\psi_j^{(\ell,k)}$ is the wavelet at $\ell$-level with $\text{supp}(\psi_j^{(\ell,k)}) \subseteq X_k^\ell$. Then

$$\langle f, \psi_j^{(\ell,k)} \rangle \leq C \rho(X_k^\ell)^{\alpha+1/2}.$$ 

On the other hand, if the inequality above holds for all wavelets $\psi_j^{(\ell,k)}$, then $f \in \mathcal{H}^{\alpha}(\mathcal{T})$.

Corollary

Let $\mathcal{T}(X)$ be a balanced tree with the upper bound $\overline{B}$. Assume $f \in H^{\alpha}(\mathcal{T})$ and $\psi_j^{(\ell,k)}$ is the wavelet at $\ell$-level with $\text{supp}(\psi_j^{(\ell,k)}) \subseteq X_k^\ell$. Then

$$\langle f, \psi_j^{(\ell,k)} \rangle \leq C \overline{B}^{(\alpha+1/2)(\ell-1)}.$$
Let $f$ be a binary classification function: $X \rightarrow \{-1, 1\}$, which is known on the labeled set $S \subset X : f(x) = y$. Then the classifier can be computed as the minimum of the following:

\[
f = \arg \min_{f \in \mathcal{H}(T)} \sum_{x \in S} \| f(x) - y \|^2 + \lambda \| W_f \|_1.
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$$f = \arg \min_{f \in \mathcal{H}(\mathcal{T})} \sum_{x \in S} \| f(x) - y \|^2 + \lambda \| W_f \|_1.$$ 

We denote by $M$ be the matrix representing the wavelet transform on $X$, by $M^T$ the inverse wavelet transform matrix. Let $S = [x_{j_1}, \cdots, x_{j_s}]$ and $P_s = [\vec{e}_{j_1}; \cdots; \vec{e}_{j_s}]$ be the landmark extraction. Then the minimization problem above becomes the following:

$$W_f = \arg \min_{W_f} (P_s M^T W_f - y)^T (P_s M^T W_f - y) + \lambda \| W_f \|_1,$$

which leads to a wavelet threshold algorithm [see Chui and Wang, 2007]
Let $g(x) = f(x) + n(x)$, where $n(x)$ is a noise on $X$. Then a simple denoising algorithm is given by

$$f = \arg \min_{f \in \mathcal{H}(T)} \sum_{x \in X} \| W_f - W_g \|_2^2 + \lambda \| W_f \|_1.$$
A set of test digits is given randomly. Only a small number of the test digits are labeled.
We select 1000 handwritten digits at random from MNIST, where 200 samples are for each of the digits 8, 3, 4, 5, 7. Digits 8 were in a class, and others are in another class.

We test the algorithm for the labeled set size $|S| = 10, 20, \cdots, 100$, that is, the label rates are from 1% to 10%.

We compare our method with three others: Laplacian Eigenvalues, Laplacian Regression, and Adaptive Threshold. They do not employ graph tree structure, but are based on manifold learning.
Experiment on 1000 samples of MNIST
We select 1500 handwritten digits at random from USPS, where 150 samples are for each of the digits from 0 to 9. Digits 2 and 5 were in a class, and others are in another class.

We test the algorithm for the labeled set size $|S| = 10, 20, \cdots, 100$, that is, the label rates are from about 0.67% to 6.67%.

We again compare our method with three others: Laplacian Eigenvalues, Laplacian Regression, and Adaptive Threshold.
Experiment on 1500 samples of USPS

Figure: Result Comparison on USPS.

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### Experiment on USPS 1500 samples: Error rates (%) of different methods.

| Method                  | $|X_0| = 10$ | $|X_0| = 100$ |
|-------------------------|-----------|-------------|
| 1-NN                    | 19.82     | 7.64        |
| SVM                     | 20.03     | 9.75        |
| MVU + 1-NN              | 14.88     | 6.09        |
| LEM + 1-NN              | 19.14     | 6.09        |
| QC + CMN                | 13.61     | 6.36        |
| Discrete Reg.           | 16.07     | 4.68        |
| TSVM                    | 25.20     | 9.77        |
| SGT                     | 25.36     | 6.80        |
| Cluster-Kernel          | 19.41     | 9.68        |
| Data-Dep. Reg.          | 17.96     | 5.10        |
| LDS                     | 17.57     | 4.96        |
| Laplacian RLS           | 18.99     | 4.68        |
| CHM (Normalized)        | 20.53     | 7.65        |
| Graph-tree Wavelets     | **8.21**  | **3.47**    |


THANK YOU !