Reducibility of metrics on the real line

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Contents

Contents

Contents

Contents

Computable categoricity

Computable model $\mathfrak{M}$ is **computably categorical** (or **autostable**) if any computable model $\mathfrak{N}$ isomorphic to $\mathfrak{M}$ is computably isomorphic to it.

Number of computable copies of $\mathfrak{M}$ that are not computably isomorphic to it is called the **computable dimension** of $\mathfrak{M}$.

Natural examples of computably categorical structures are quite rare.

Example 1.1

$\langle \mathbb{Q}, \leqslant \rangle$ *is computably categorical.*

Theorem 1.1 (A. Maltsev)

Any finitely generated computable model is computably categorical.

Computable categoricity

Theorem 1.2

$\langle \mathbb{N}, \leqslant \rangle$ *is not computably categorical.*

Theorem 1.3 (A. Fröhlich, J. Shepherdson)

There exists a computable field that is not computably categorical.

Theorem 1.4 (A. Maltsev)

There exists a computable abelian group that is not computably categorical.

Computable categoricity

Theorem 1.5 (A. Nurtazin)

A decidable structure either has computable dimension 1 *or* ω*.*

Theorem 1.6 (A. Nurtazin; G. Metakides, A. Nerode; S. Goncharov; S. Goncharov, V. Dzgoev; P. LaRoche; J. Remmel)

Structures of the following classes either have computable dimension 1 *or* ω*: algebraically closed fields; real closed fields; abelian groups; linear orderings; Boolean algebras;* Δ^0_2*-categorical structures.*

Theorem 1.7 (S. Goncharov)

For all $n > 1$*, there exists a computable structure of computable dimension* n*.*

$\mathbf{d}$-computable categoricity

Computable model $\mathfrak{M}$ is **$\mathbf{d}$-computably categorical** if any computable $\mathfrak{N}$ isomorphic to $\mathfrak{M}$ is $\mathbf{d}$-computably isomorphic to it.

The **autostability spectrum** of $\mathfrak{M}$ is the set

$$AutSpec(\mathfrak{M}) = \{\mathbf{d} \mid \mathfrak{M} \textit{ is } \mathbf{d}\textit{-computably categorical}\}.$$

Theorem 1.8 (D. Hirschfeldt, B. Khoussainov, R. Shore, A. Slinko)

For any computable model $\mathfrak{M}$, there exist computable models of the following classes that have the same autostability spectrum as $\mathfrak{M}$: directed graphs; symmetric, irreflexive graphs; partial orderings; lattices; rings (with zero-divisors); integral domains; commutative semigroups; 2-step nilpotent groups.

Computability theory in analysis

A. Turing, *On computable numbers, with an application to the "Entscheidungsproblem"*, 1936-37:

Definition 1.1

*Real number is **computable** if its decimal expansion is computable.*

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As it turns out, addition and multiplication are not computable with respect to decimal representation of the reals.

Different representations (by Dedekind cuts, left or right Dedekind cuts, quickly converging sequences of rationals, continued fractions) lead to different notions of effectivity for real numbers.

Computability theory in analysis

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Computability theory in analysis

Applications of computability theory in analysis have been approached from various points of view.

- Banach and Mazur's computability
- Approach of Moschovakis
- Russian constructive school: Ceitin, Kushner, Shanin etc.
- Computability in Banach spaces by Pour-El and Richards
- Representation approach of Kreitz and Weihrauch

Pour-El and Richards's approach

Computability in Analysis and Physics, 1980

Pour-El and Richards studied the question of uniqueness of "computability structure" (the system of all computable sequences) in separable Banach space up to computable isometry.

Theorem 1.9

- *All computability structures in Hilbert space are pairwise computably isometric*
- *However, there exists a structure in the space* l_1 *that is not computably isometric to the standard structure of this space*

Further results on uniqueness of computable structure

- A. Melnikov, *Computably Isometric Spaces*, 2013

  l_1 is not computably categorical as a metric space

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- Z. Iljazović, Isometries and Computability Structures, 2010

  All countable dense structures in an effectively compact computable metric space are pairwise computably isometric

Further results on uniqueness of computable structure

- A. Melnikov, K. Ng, *Computable structures and operations on the space of continuous functions*, 2015

  $C[0,1]$ (as a metric space) has computable dimension ω.

  $C[0,1]$ (as a Banach space) is not computably categorical.

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- T. McNicholl, D. Stull, *The isometry degree of a computable copy of* l_p, 2016

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The main question

Another approach: for the space of real numbers, fix rationals as the dense substructure and examine different metrics that induce the standard topology.

The main question can be stated as follows:

Is it possible to construct a metric on $\mathbb{R}$ such that:

- It is computable
- It induces the usual topology on $\mathbb{R}$
- $\mathbb{R}$ equipped with this metric is computably inequivalent to the standard real line?

The main question

Another approach: for the space of real numbers, fix rationals as the dense substructure and examine different metrics that induce the standard topology.

- There are infinitely many such metrics
- There are infinitely many metrics such that copies of $\mathbb{R}$ equipped with them are not computably homeomorphic to each other

Definition 2.1

A ***computable functional*** *is a function* $\Phi\colon \omega^\omega \to \omega^\omega$ *such that for some oracle computable function* Φ_e

$$\Phi(f) = g \text{ iff } \Phi_e^f(n) = g(n)$$

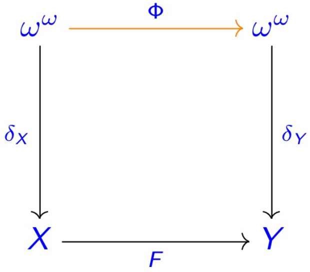
Definition 2.2

A ***representation*** *of a set* X *is a partial surjection* $\delta\colon \omega^\omega \to X$.

Definition 2.3

A partial function $F\colon X \to Y$ *is* (δ_X, δ_Y)*-**computable** if there exists a computable functional* Φ *such that*

$$F\delta_X(f) = \delta_Y\Phi(f) \text{ for } f \in dom(F\delta_X).$$



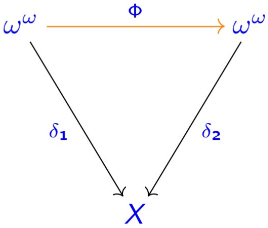
Reducibility of representations

K. Weihrauch, *Computable analysis. An Introduction*, 2000

Definition 2.4

Let δ_1, δ_2 be representations. δ_1 is ***computably reducible*** *to δ_2 ($\delta_1 \leq_c \delta_2$) if there exists a computable functional Φ such that*

$$\delta_1(f) = \delta_2\Phi(f) \text{ for } f \in dom(\delta_1)$$



or, equivalently, if the identity function id_X is (δ_1, δ_2)-computable.

Reducibility of representations

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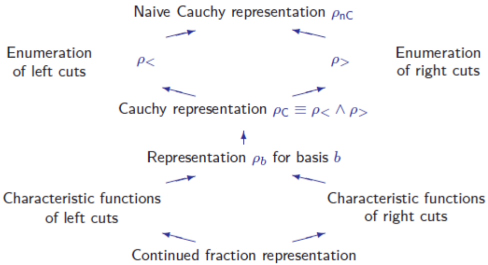

Picture ©V. Brattka

Cauchy representations

Definition 2.5

Let (X, ρ) be a complete separable metric space with a dense countable subset $W \subseteq X$, $W = (w_n)_{n \in \omega}$.

*The space $\mathbf{X} = (X, \rho, W)$ is called an **effective metric space**.*

*If the distance function $\rho(w_n, w_m) \in \mathbb{R}_c$ is computable in n and m, effective space $\mathbf{X}$ and metric ρ are called **computable**.*

Cauchy representations

Definition 2.6

Cauchy representation $\delta_\rho\colon \omega^\omega \to X$ *is defined as follows: for* $x \in X$ *and* $f \in \omega^\omega$ *we say that* f *is a* ***Cauchy name*** *for* x*, or* $\delta_\rho(f) = x$*, if*

$$w_{f(n)} \to x \text{ and } \rho(w_{f(n)}, w_{f(m)}) \leqslant 2^{-n} \text{ for } m > n.$$

Let (X, ρ_1, W) and (X, ρ_2, W) be effective metric spaces. We say $\rho_1 \leq_c \rho_2$ if $\delta_{\rho_1} \leq_c \delta_{\rho_2}$.

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Lemma 2.1

If $\exists M > 0\ \forall x, y \in X$

$$\rho_2(x, y) \leqslant M \cdot \rho_1(x, y)$$

(id_X is Lipschitz continuous w.r.t. ρ_2 and ρ_1), then $\rho_1 \leq_c \rho_2$.

Convex metrics

Convex metric space is a space (X, ρ) for any two points of which there exists an exact midpoint between them.

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All convex computable metrics on $\mathbb{R}$ are c-equivalent.

Proof.

Note that convex metrics respect the usual ordering of $\mathbb{R}$. Let ρ_1, ρ_2 be convex.
Given any sequence $(x_n)_{n\in\omega}$ of rationals effectively converging to $x \in \mathbb{R}$ w.r.t. ρ_1, we can construct a sequence $(r_n, s_n)_{n\in\omega}$ of rational intervals converging to x. Then $\rho_2(r_n, s_n)$ is an upper bound for $\rho_2(r_n, x)$. Measuring this distance, we find a subsequence of $(r_n)_{n\in\omega}$ that converges to x effectively. □

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Measuring this distance, we find a subsequence of $(r_n)_{n\in\omega}$ that converges to x effectively. □

So, in order to construct computably inequivalent metrics, we should consider non-convex metrics

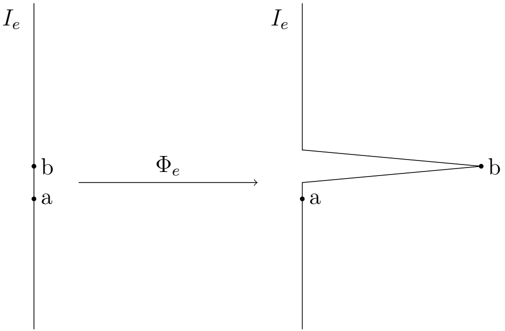
Inequivalent metrics

Theorem 3.2

There exists a computable metric $\rho <_c \rho_{\mathbb{R}}$.

Proof idea

We diagonalize against Φ_e on a distinct interval I_e in $\mathbb{R}$



1. Pick an element $a \in I_e$, $\delta_{\rho_\mathbb{R}}$-name f for a and compute $\Phi_e(\bar{f})$, $\bar{f}$ initial segment of f.

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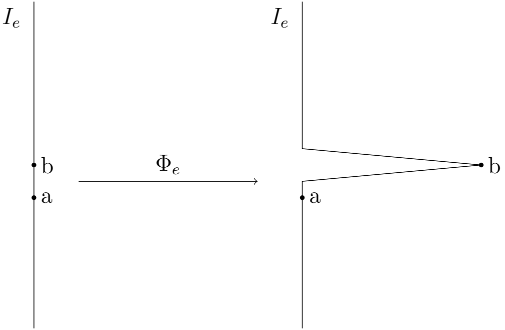

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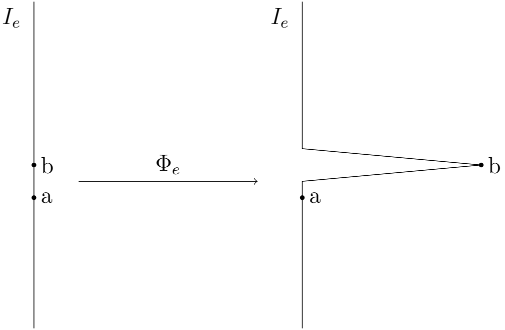

1. Pick an element $a \in I_e$, $\delta_{\rho_{\mathbb{R}}}$-name f for a and compute $\Phi_e(\bar{f})$, $\bar{f}$ initial segment of f.
2. Pick b close to a so $\bar{f}$ is initial segment of a name for b as well.
3. Make a continuous "peak" so that $\rho(a, b)$ is large enough.

Iterating the construction

By properly combining intervals and peaks, it is also possible to define countable sequences of incomparable metrics and construct metrics that lie even lower in the ordering $\leq_c$.

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Theorem 3.3

$\omega^{<\omega}$ is isomorphically embeddable into the ordering $\leq_c$ of computable metrics.

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Theorem 3.5

The class of c-inequivalent computable metrics is effectively infinite (i.e. for any computable sequence ρ_i of computable metrics we can construct metric ρ such that $\rho \not\equiv_c \rho_i$ for all i).

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However, it is not known how to construct metrics higher than the given metric in $\leq_c$.

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Ordering $\leq_c$ of computable metrics is downward closed.

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Lemma 3.1

Ordering $\leq_c$ *of computable metrics is downward closed.*

Proof.

For computable ρ_1, ρ_2 *their maximum* $\rho = \max\{\rho_1, \rho_2\}$ *is a computable metric that induces the same topology and for all* x, y

$$\rho_1(x, y), \rho_2(x, y) \leqslant \rho(x, y),$$

thus $\rho \leq_c \rho_1, \rho_2$. □

Reducibility $\leq_{ch}$

Let $\delta\colon \omega^\omega \to X$ be a representation. The **final topology** of δ is the finest topology τ_δ of X with respect to which δ is continuous.

Definition 3.1

Let representations X δ_1 and δ_2 have the same final topology. We say that $\delta_1 \leq_{ch} \delta_2$ if there exists a (δ_1, δ_2)-computable autohomeomorphism of X.

Lemma 3.2

If $\delta_1 \leq_c \delta_2$, then $\delta_1 \leq_{ch} \delta_2$.

Proof.

$\delta_1 \leq_c \delta_2$ *means that id_X is a (δ_1, δ_2)-computable homeomorphism.* □

Metrics that admit no computable homeomorphism

Theorem 3.6

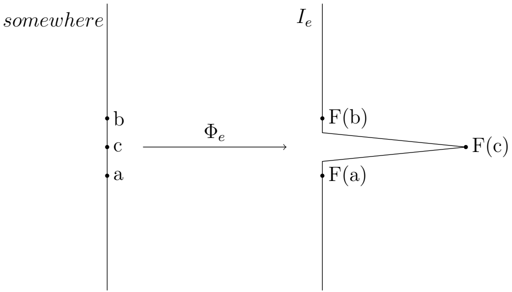
There exists a computable metric $\rho <_{ch} \rho_{\mathbb{R}}$.

Informally:

- There exists a computable homeomorphism $(\mathbb{R}, \rho) \to (\mathbb{R}, \rho_{\mathbb{R}})$
- There is no computable homeomorphism $(\mathbb{R}, \rho_{\mathbb{R}}) \to (\mathbb{R}, \rho)$

Proof idea

Now we try to capture arbitrary rationals' images inside I_e



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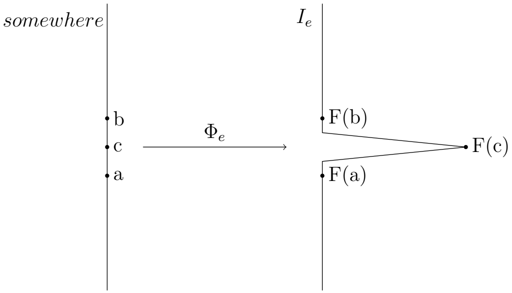

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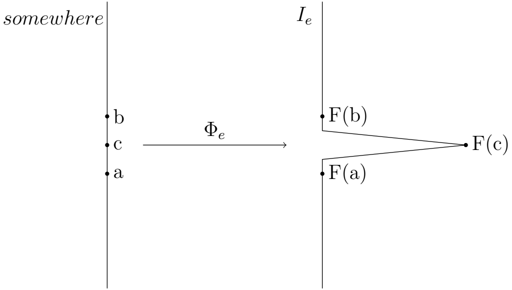

1. Search for some rational a and b on the whole real line that are mapped to I_e.
2. Make sure that a and b are mapped to different points in I_e.
3. Some c between a and b will then be mapped to the top of the peak.

Metrics that admit no computable homeomorphism

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Real line (with rationals as a dense subset) has computable dimension ω.

Construction in more general case

When establishing effective infinity, we had to deal with sequences of arbitrary computable metrics on $\mathbb{R}$ that may have very little in common with the standard metric from geometrical point of view. Hence question: do our results extend to a whole class of “real-like” metric spaces?

Construction in more general case

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An easy idea is to extend the results to path connected spaces. Given a path between two points, one can effectively choose a countable sequence of disjoint open balls along it and run similar diagonal construction. Extra attention should be paid to making sure that topology is preserved by the constructed metric.

Construction in more general case

Hypothesis 1

For any path connected computable metric space (X, ρ, W) *the following holds:*

1. $\omega^{<\omega}$ *is isomorphically embeddable into the ordering* $\leq_c$ *of computable metrics that induce the same topology as* ρ*.*
2. *Any constructive ordinal is isomorphically embeddable into the ordering* $\leq_c$ *of computable metrics that induce the same topology as* ρ*.*
3. *The class of* c*-inequivalent computable metrics is effectively infinite.*

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3. *The class of* c*-inequivalent computable metrics is effectively infinite.*

Hypothesis 2

If computable metric space (X, ρ, W) *is convex, then there exists a countable anti-chain* $(\rho_i)_{i\in\omega}$ *of computable metrics that induce the same topology as* ρ, *are incomparable to each other w.r.t* $\leq_{ch}$ *and* c*-reducible to* ρ.

Going beyond path connectedness

Path connectedness is (almost) only needed to give us a sequence of balls that can be used in diagonal construction.

Does the construction work in connected spaces?

Other classes of spaces?

Going beyond convexity

Convexity is a very unnatural condition for existence of *ch*-inequivalent metrics.

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Does there exist a construction independent from convexity?

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Convexity is a very unnatural condition for existence of *ch*-inequivalent metrics.

Does there exist a construction independent from convexity?

Can chains in the ordering $\leq_{ch}$ be constructed?