Section 1

The ITTM model
An infinite-time Turing machine is a Turing machine with three tapes whose cells are indexed by natural numbers:

- The input tape
- The output tape
- The working tape

It behaves like a standard Turing machine at successor steps of computation.
Infinite time Turing machines

At limit steps of computation:

- The head goes back to the first cell.
- The machine goes into a “limit” state.
- The value of each cell equals the lim inf of the values at previous stages of computation.
What is the equivalent of computable for an ITTM?

**Definition**

A real $X$ is **writable** if there in an ITTM $M$ such that:

$M(0) \downarrow [\alpha] = X$ for some ordinal $\alpha$.

- **$M$ starts with $0$ on its input tape**
- **$M$ enters its halting state at step $\alpha + 1$**
- **$X$ is on the output tape when $M$ halts**

$M(0) \downarrow [\alpha] = X$
Decidable classes

Which reals are writable?

Definition
A class of real $\mathcal{A}$ is **decidable** if there is an ITTM $M$ such that $M(X) \downarrow = 1$ if $X \in \mathcal{A}$ and $M(X) \downarrow = 0$ if $X \notin \mathcal{A}$.

Proposition (Hamkins, Lewis)
The class of reals coding for a well-order (with the code $X(\langle n, m \rangle) = 1$ iff $n < m$) is decidable.
Proposition (Hamkins, Lewis)

The class of reals coding for a well-order (with the code $X(\langle n, m \rangle) = 1$ iff $n < m$) is decidable.

The algorithm is as follow, where $<$ is the order coded by $X$:

Algorithm to decide well-orders

\begin{algorithm}
\textbf{while} $<$ is not empty \textbf{do}
\begin{algorithmic}
  \State Look for the smallest element $a$ of $<$ (coded by $X$)
  \If {there is no smallest element}
  \State write 0 and halts
  \Else
  \State remove $a$ from the support of $<$
  \EndIf
\EndWhile
\State When $<$ is empty, write 1 and halts.
\end{algorithmic}
\end{algorithm}
Decide well-orders

How to find the smallest element?

Algorithm to find the smallest element

Write 1 on the first cell. Set the current element $c = +\infty$

if state is successor then
  if there exists $a < c$ then
    Update $c = a$
    Flip the first cell to 0 and then back to 1
  end
else
  if the first cell is 0 then
    There is no smallest element
  else
    $c$ is the smallest element
  end
end
Decidable and writable sets

Proposition (Hamkins, Lewis)
The class of reals coding for a well-order (with the code $X(\langle n, m \rangle) = 1$ iff $n < m$) is decidable.

Corollary (Hamkins, Lewis)
Every $\Pi^1_1$ set of reals is decidable.

Corollary (Hamkins, Lewis)
Every $\Pi^1_1$ set of integers is writable.
Computational power of ITTM

$\omega_1^{ck}$ steps of computations are enough to write any $\Pi^1_1$ set of integers. But there is no bound in the ordinal step of computation an ITTM can use.

Using a program that writes Kleene’s $O$, we can design a program which writes the double hyperjump $O^O$ and then $O^{(O^O)}$ and so on.

Where does it stop?

Proposition (Hamkins, Lewis)

Whatever an ITTM does, it does it before stage $\omega_1$. 
Computational power of ITTM

**Proposition (Hamkins, Lewis)**

Whatever an ITTM does, it does it before stage $\omega_1$.

The configuration of an ITTM is given by:

1. Its tapes
2. Its state
3. The position of the head.

Let $C(\alpha) \in 2^\omega$ be a canonical encoding of the tapes of an ITTM at stage $\alpha$.

There must be some *limit ordinal* $\alpha < \omega_1$ such that $C(\alpha) = C(\omega_1)$. The full configuration of the machine at step $\omega_1$ is then the same than the one step $\alpha$. 

Computational power of ITTM

\[ \omega_1 \]

\[ \sup_n \alpha_n^+ \]

\[ \alpha_2^+ > \alpha_1^+ \]

\[ \alpha_1^+ > \alpha_0 \]

\[ \alpha_0 \]

\[ \alpha_0 : \text{The smallest ordinal such that every cell converging at step } \omega_1 \text{ (in green) will never change pass that point.} \]

\[ \alpha_{n+1}^+ : \text{The smallest ordinal } > \alpha_n^+ \text{ such that the } n+1 \text{ non-converging cells (in red) change value at least once in the interval } [\alpha_n^+, \alpha_{n+1}^+] \]
Beyond the writable ordinals

**Definition (Hamkins, Lewis)**

An ordinal $\alpha$ is **writable** if there is an ITTM which writes an encoding of a well-order of $\omega$ with order-type $\alpha$.

**Proposition (Hamkins, Lewis)**

The writables are all initial segments of the ordinals.

**Definition (Hamkins, Lewis)**

Let $\lambda$ be the supremum of the writable ordinals.

**Proposition (Hamkins, Lewis)**

There is an ITTM which writes $\lambda$ on its output tape, then leave the output tape unchanged without ever halting.
Beyond the writable ordinals

Proposition (Hamkins, Lewis)
There is a universal ITTM $U$ which runs simultaneously all the ITTM computations $P_e(0)$ for every $e \in \omega$.

Algorithm to eventually write $\lambda$

```latex
\textbf{for} every stage $s$ \textbf{do}
    \begin{itemize}
    \item Run the universal machine $U$ for one step.
    \item Compute the sum $\alpha_s$ of all ordinals which are on the output tapes of programs simulated by $U[s]$ and which have terminated.
    \item Write $\alpha_s$ on the output tape.
    \end{itemize}
\textbf{end}
```

Let $s$ be the smallest stage such that every halting ITTM have halted by stage $s$ in the simulation $U$.

1. We clearly have $\alpha_s \geq \lambda$.
2. We clearly have that $\alpha_t = \alpha_s$ for every $s \geq t$. 
Beyond the eventually writable ordinals

Definition (Hamkins, Lewis)
A real is **eventually writable** if there is an ITTM and a step $\alpha$ such that for every $\beta \geq \alpha$, the real is on the output tape at step $\beta$.

Proposition (Hamkins, Lewis)
The eventually writable ordinals are an initial segment of the ordinals.

Definition (Hamkins, Lewis)
Let $\zeta$ be the supremum of the eventually writable ordinals.

Proposition (Hamkins, Lewis)
There is an ITTM which at some point writes $\zeta$ on its output tape.
Beyond the eventually writable ordinals

Algorithm to accidentally write $\zeta$

for every stage $s$ do
    Run the universal machine $U$ for one step.
    Compute the sum $\alpha_s$ of all ordinals which are on the output
tapes of programs simulated by $U[s]$.
    Write $\alpha_s$ on the output tape.
end

Let $s$ be the smallest stage such that every ITTM writing an eventually writable ordinal, have done so by stage $s$ in the simulation $U$. We clearly have $\alpha_s \geq \zeta$. 
Beyond the eventually writable ordinals

**Definition (Hamkins, Lewis)**

A real is **accidentally writable** if there in an ITTM and a step $\alpha$ such that the real is on the output tape at step $\alpha$.

**Proposition (Hamkins, Lewis)**

The accidentally writables are all initial segments of the ordinals.

**Definition (Hamkins, Lewis)**

Let $\Sigma$ be the supremum of the accidentally writables.

**Proposition (Hamkins, Lewis)**

We have $\lambda < \zeta < \Sigma$. 
ITTM and constructibility

Section 2

ITTM and constructibility
The constructibles

**Definition (Godel)**

The **constructible universe** is defined by induction over the ordinals as follows:

\[
\begin{align*}
L_\emptyset &= \emptyset \\
L_{\alpha^+} &= \{ X \subseteq L_\alpha : X \text{ is f.o. definable with param. in } L_\alpha \} \\
L_{\sup_n \alpha_n} &= \bigcup_n L_{\alpha_n}
\end{align*}
\]

**Theorem (Hamkins, Lewis)**

- If \( \alpha \) is writable and \( X \in 2^\omega \cap L_\alpha \) then \( X \) is writable.
- If \( \alpha \) is eventually writable and \( X \in 2^\omega \cap L_\alpha \) then \( X \) is eventually writable.
- If \( \alpha \) is accidentally writable and \( X \in 2^\omega \cap L_\alpha \) then \( X \) is accidentally writable.
### The admissibles

**Definition (Admissibility)**

An ordinal $\alpha$ is **admissible** if $L_\alpha$ is a model of $\Sigma_1$-replacement. Formally for any $\Sigma_1$ formula $\Phi$ with parameters and any $N \in L_\alpha$ we must have:

\[
\begin{align*}
L_\alpha & \models \forall n \in N \exists z \Phi(n, z) \\
\rightarrow \quad L_\alpha & \models \exists Z \forall n \in N \exists z \in Z \Phi(n, z)
\end{align*}
\]

$\omega, \omega_{1}^{ck}, \omega_{2}^{ck}, \omega_{3}^{ck}, etc...$ are the first admissible ordinals.

Consider the formula $\exists n \forall k < n \exists m A(n, k, m)$ (with $A \Delta_0$). The formula is $\Sigma_1$ : This is because if for every $k < n$, there exists a witness $m_k$ such that $A(n, k, m_k)$, then $\sup_k m_k$ is still finite.

The admissible are the sets for which this property is still true.
The admissibles

Proposition (Hamkins, Lewis)
The ordinals $\lambda$ and $\zeta$ are admissible.

Suppose that for some $N \in L_\lambda$ and a $\Sigma_1$ formula $\Phi$ we have:

$$L_\lambda \models \forall n \in N \exists z \Phi(n, z)$$

We define the following ITTM:

Algorithm to show $\lambda$ admissible

- Write a code for $N$
  - for every $n \in N$ do
    - Look for the first writable $\alpha_n$ such that $L_{\alpha_n} \models \exists z \Phi(n, z)$
    - Write $\alpha_n$ somewhere.
  - end

- Write $\sup_{n \in N} \alpha_n$
The admissibles

**Proposition (Hamkins, Lewis)**

The ordinals $\lambda$ is the $\lambda$-th admissible.
The ordinals $\zeta$ is the $\zeta$-th admissible.

Suppose $\lambda$ is the $\alpha$-th admissible for $\alpha < \lambda$.

**Algorithm to show $\lambda$ is the $\lambda$-th admissible**

Write $\alpha$

while $\alpha > 0$ do

Look for the smallest element $e$ of $\alpha$ and remove it from $\alpha$

Look for the next admissible writable ordinal and write it to the $e$-th tape

end

Write the smallest admissible greater than all the one written previously.
How big is $\lambda$

Definition
An ordinal is **recursively inaccessible** if it is admissible and limit of admissible.

Proposition (Hamkins, Lewis)
The ordinals $\lambda$ is the $\lambda$-th recursively admissible.
The ordinals $\zeta$ is the $\zeta$-th recursively admissible.

Definition
An ordinal is **meta-recursively inaccessible** if it is admissible and a limit of recursively inaccessible.

Proposition (Hamkins, Lewis)
The ordinals $\lambda$ is the $\lambda$-th meta recursively admissible.
The ordinals $\zeta$ is the $\zeta$-th meta recursively admissible.
The clockable ordinals

Section 3

The clockable ordinals
The clockable ordinals

Another notion will help us to understand better $\lambda$, $\zeta$ and $\Sigma$

**Definition (Hamkins, Lewis)**

An ordinal $\alpha$ is **clockable** if there is an ITTM which halts at stage $\alpha$ (at stage $\alpha$ it decides to go into the halting state).

**What is the supremum of the clockable ordinals?**

**Definition (Hamkins, Lewis)**

Let $\gamma$ be the supremum of the clockable ordinals.

**Proposition (Hamkins, Lewis)**

We have $\lambda \leq \gamma$. 
The clockable ordinals

**Proposition (Hamkins, Lewis)**

We have $\lambda \leq \gamma$.  

Suppose the ITTM $M$ writes $\alpha$. Then one can easily create an ITTM which does the following:

---

**Algorithm to countdown $\alpha$**

Use $M$ to write $\alpha$

**while** $\alpha > 0$ **do**

1. Find the smallest element of $\alpha$ and remove it from $\alpha$.

**end**

Enter the halting state.

---

It is easy to see that the above algorithm takes at least $\alpha$ step before it ends.
Understanding the clockables

Theorem (Hamkins, Lewis)

The clockable ordinals are not an initial segment of the ordinals: If $\alpha$ is admissible then no ITTM halts in $\alpha$ steps.

For $\alpha$ limit to be clockable we need for some $i \in \{0, 1\}$ to have both :

1. A transition rule of the form: $(\text{limit, } i) \rightarrow \text{halt}$

2. The first cell to contain $i$ at step $\alpha$

If $\{C_i(\gamma)\}_{\gamma < \alpha}$ converges we have a limit $\beta < \alpha$ s.t. $C_i(\beta) = C_i(\alpha)$

$\rightarrow$ We have (1) and (2) for $\beta < \alpha$

If $\{C_i(\gamma)\}_{\gamma < \alpha}$ diverges, let:

$f(n + 1) =$ the smallest $\alpha > f(n)$ s.t. $C_0(\beta)$ changes for $\beta \in [f(n), \alpha]$

$\rightarrow$ By admissibility $\sup_n f(n) < \alpha$ and we have (1) and (2) for $\sup_n f(n)$

In both cases the machine stopped before stage $\alpha$. 
Understanding the clockables

Definition (Hamkins, Lewis)

A **gap of size** $\alpha$ in the clockable ordinals is an interval of ordinals $[\alpha_0, \alpha_0 + \alpha]$ such that no ITTM halts in this interval, but some halt after that.

Theorem (Hamkins, Lewis)

For any writable $\alpha$, there is a gap of size at least $\alpha$ in the clockable ordinals.
Algorithm to witness gap of size $\alpha$

Run the universal ITTM

```
while true do
  if a new ordinal $\alpha_0$ is written on a tape then
    if no ITTM halts in the interval $[\alpha_0, \alpha_0 + \alpha]$ then
      Write $\alpha_0 + \alpha$ and halt.
    end
  end
end
```

Note that if $\alpha$ is writable then $\lambda + \alpha < \zeta < \Sigma$. Suppose there is no gap of size $\alpha$.

→ Then the algorithm will at some point:

1. Eventually write $\lambda$ and will see that no ITTM halts in $[\lambda, \lambda + \alpha]$

2. Write $\lambda + \alpha$ and halts

This is a contradiction.
Understanding $\lambda, \zeta, \Sigma$

**Lemma (Welch)**

Let $i \in \omega$. If the sequence $\{C_i(\alpha)\}_{\alpha<\lambda}$ converges, then for every $\alpha \in [\lambda, \Sigma]$ we have $C_i(\alpha) = C_i(\lambda)$.

Suppose w.l.o.g. that $\{C_i(\alpha)\}_{\alpha<\lambda}$ converges to 0. Let $\beta$ be the smallest such that for all $\alpha \in [\beta, \lambda]$ we have $C_i(\alpha) = 0$.

**Algorithm**

```
for every $\alpha > \beta$ written by $U$ do
    Simulate another run of $U$ for $\alpha$ steps
    if $C_i(\gamma) = 1$ for $\gamma \in [\beta, \alpha]$ then
        Write $\alpha$ and halt.
    end
end
```

Suppose there is an accidentally writable ordinal $\alpha > \beta$ s.t. $C_i(\alpha) = 1$. Then $U$ will write such an ordinal at some point, and the above program will then write $\alpha > \lambda$ and halt. This is a contradiction.
Theorem (Welch)

The whole state of an ITTM at step $\zeta$ is the same than its state at step $\Sigma$. In particular, it enters an infinite loop at stage $\zeta$.

The theorem follows from the two following lemmas:

Lemma (Welch)

Let $i \in \omega$. If the sequence $\{C_i(\alpha)\}_{\alpha < \zeta}$ converges, then for every $\alpha \in [\zeta, \Sigma]$ we have $C_i(\alpha) = C_i(\zeta)$.

Lemma (Welch)

Let $i \in \omega$. If the sequence $\{C_i(\alpha)\}_{\alpha < \zeta}$ diverges, then the sequence $\{C_i(\alpha)\}_{\alpha < \Sigma}$ diverges.
Understanding $\lambda, \zeta, \Sigma$

Suppose w.l.o.g. that $\{C_i(\alpha)\}_{\alpha<\zeta}$ converges to 0.
Let $\beta$ be the smallest such that for all $\alpha \in [\beta, \zeta]$ we have $C_i(\alpha) = 0$.
The ordinal $\beta$ is eventually writable through different versions $\{\beta_s\}_{s \in \text{ORD}}$

Algorithm

\begin{verbatim}
for every $s$ and every $\alpha > \beta_s$ written by $U$ do
    Simulate another run of $U$ for $\alpha$ steps
    if $C_i(\gamma) = 1$ for $\gamma \in [\beta_s, \alpha]$ and $\beta_s$ has changed then
        Write $\alpha$ on the output tape.
end
\end{verbatim}

Suppose there is an accidentally writable ordinal $\alpha > \beta$ s.t. $C_i(\alpha) = 1$.
Then some ordinal $\alpha' \geq \alpha$ will be written at some stage at which $\beta_s$ has stabilized.
Thus the above program will then eventually write some $\alpha' > \zeta$.
This is a contradiction.
Understanding $\lambda, \zeta, \Sigma$

Suppose $\{C_i(\alpha)\}_{\alpha<\Sigma}$ converges.

---

**Algorithm**

Set $\beta = 0$

for every $\alpha > \beta$ written by $U$ do

| Simulate another run of $U$ for $\alpha$ steps |
| if $C_i(\gamma)$ changes for $\gamma \in [\beta, \alpha]$ then |
| Let $\beta = \alpha$ |
| Write $\alpha$ |

end

---

The algorithm will eventually write some ordinal $\alpha$ s.t. $\{C_i(\gamma)\}$ does not change for $\gamma \in [\alpha, \Sigma]$. But then $\alpha$ is eventually writable and $\{C_i(\alpha)\}_{\alpha<\zeta}$ converges.
Understanding $\lambda, \zeta, \Sigma$

**Theorem (Welch)**
The whole state of an ITTM at step $\zeta$ is the same than its state at step $\Sigma$. In particular, it enters an infinite loop at stage $\zeta$.

**Corollary (Welch)**
$\lambda$ is the supremum of the clockable ordinals.

Indeed, suppose that we have $M(0) \downarrow [\alpha]$ for some $M$ and $\alpha$ accidentally writable. Then we can run $M(0)[\beta]$ for every $\beta$ accidentally writable until we find one for which $M$ halts, and then write $\beta$. Thus $\alpha$ must be writable.

Suppose now that $M(0) \uparrow [\Sigma]$. Then $M$ will never halt. Thus if $M$ halts, it halts at a writable step.
Theorem (Welch)
The whole state of an ITTM at step $\zeta$ is the same than its state at step $\Sigma$. In particular, it enters an infinite loop at stage $\zeta$. 

Corollary (Welch)
- The writable reals are exactly the reals of $L_\lambda$.
- The eventually writable reals are exactly the reals of $L_\zeta$.
- The accidentally writable reals are exactly the reals of $L_\Sigma$.

We can construct every successive configurations of a running ITTM. Also to compute a writable reals, there are less than $\lambda$ steps of computation and then less than $\lambda$ steps of construction. Thus every writable real is in $L_\lambda$.

The argument is similar for $\zeta$ and $\Sigma$. 
Understanding $\lambda, \zeta, \Sigma$

**Definition**

Let $\alpha \leq \beta$. We say that $L_\alpha$ is $n$-stable in $L_\beta$ and write $L_\alpha <_n L_\beta$ if

$$L_\alpha \models \Phi \iff L_\beta \models \Phi$$

For every $\Sigma_n$ formula $\Phi$ with parameters in $L_\alpha$.

**Theorem (Welch)**

$(\lambda, \zeta, \Sigma)$ is the lexicographically smallest triplet such that:

$$L_\lambda <_1 L_\zeta <_2 L_\Sigma$$
Understanding $\lambda, \zeta, \Sigma$

**Theorem (Welch)**

The ordinal $\Sigma$ is not admissible.

To see this, we define the following function $f : \omega \rightarrow \Sigma :$

$$f(0) = \zeta$$
$$f(n) = \text{the smallest } \alpha \text{ s.t. } C(\alpha) \upharpoonright n = C(\zeta) \upharpoonright n$$

It is not very hard to show that we must have $\sup_n f(n) = \Sigma$

**Theorem (Welch)**

The ordinal $\Sigma$ is a limit of admissible.

Otherwise, if $\alpha$ is the greatest admissible smaller than $\Sigma$, one could compute $\Sigma \leq \omega_\alpha^\alpha$. 
Section 4

ITTM and randomness
ITTM and randomness

Definition (Carl, Schlicht)

$X$ is $\alpha$-random if $X$ is in no set whose Borel code is in $L_\alpha$.

Definition

An open set $U$ is $\alpha$-c.e. if $U = \bigcap_{\sigma \in A}[\sigma]$ for a set $A \subseteq 2^{<\omega}$ such that:

$$\sigma \in A \leftrightarrow L_\alpha \models \Phi(\sigma)$$

for some $\Sigma_1$ formula $\Phi$ with parameters in $L_\alpha$.

Definition (Carl, Schlicht)

$X$ is $\alpha$-ML-random if $X$ is in no set uniform intersection $\bigcap_n U_n$ of $\alpha$-c.e. open set, with $\lambda(U_n) \leq 2^{-n}$. 
Projectibles and ML-randomness

**Definition**

We say that $\alpha$ is **projectible** into $\beta < \alpha$ if there is an injective function $f : \alpha \to \beta$ that is $\Sigma_1$-definable in $L_\alpha$.

The least $\beta$ such that $\alpha$ is projectible into $\beta$ is called the **projectum** of $\alpha$ and denoted by $\alpha^*$.

**Theorem (Angles d’Auriac, Monin)**

The following are equivalent for $\alpha$ limit such that $L_\alpha \models$ everything is countable:

- $\alpha$ is projectible into $\omega$.
- There is a universal $\alpha$-ML-test.
- $\alpha$-ML-randomness is strictly stronger than $\alpha$-randomness.
\textbf{Theorem}

The ordinal $\lambda$ is projectible into $\omega$ without using any parameters.

Each writable ordinal can be effectively assigned to the code of the ITTM writing it.

\textbf{Corollary}

Most work in $\Delta^1_1$ and $\Pi^1_1$-ML-randomness still work with $\lambda$-ML-randomness and $\lambda$-randomness. In particular $\lambda$-ML-randomness is strictly weaker than $\lambda$-randomness.
The ITTM model ITTM and constructibility The clockable ordinals ITTM and randomness

ζ-ML-randomness

Theorem
The ordinal $\zeta$ is not projectible into $\omega$.

Suppose that an eventually writable parameter $\alpha$ can be used to have a projectum $f : \zeta \rightarrow \omega$. Then every eventually writable ordinal become writable using $\alpha$. Then $\zeta$ becomes eventually writable using $\alpha$. But then $\zeta$ is eventually writable.

Corollary
$\zeta$-randomness coincides with $\zeta$-ML-randomness. An analogue of $\Omega$ for $\zeta$-randomness does not exists.
# ζ-ML-randomness

## Theorem

The ordinal $\zeta$ is not projectible into $\omega$.

## Corollary

For many writable ordinals $\alpha$ we have that $\alpha$-randomness coincides with $\alpha$-ML-randomness.

\[
L_{\Sigma} \models \exists \alpha \text{ not projectible into } \omega
\]

By the fact that $L_{\lambda} <_1 L_{\Sigma}$ we must have:

\[
L_{\lambda} \models \exists \alpha \text{ not projectible into } \omega
\]
The ordinal $\Sigma$ is projectible into $\omega$, using $\zeta$ as a parameter.

We can use the fact that $(\zeta, \Sigma)$ is the least pair such that:
$C(\zeta) = C(\Sigma)$, with the function:

\[
\begin{align*}
f(0) &= \zeta \\
f(n) &= \text{the smallest } \alpha \text{ s.t. } C(\alpha) \uparrow_n = C(\zeta) \uparrow_n
\end{align*}
\]

Every ordinal $f(n)$ is then $\Sigma_1$-definable with $\zeta$ as a parameter.

As $L_{\Sigma} \models \text{"everything is countable"}$, it follows that every ordinal smaller than $f(n)$ for some $n$ is $\Sigma_1$-definable with $\zeta$ as a parameter.

As $\sup_n f(n) = \Sigma$, it follows that every accidentally writable is $\Sigma_1$-definable with $\zeta$ as a parameter.

The projectum is then a code for the formula defining each ordinal.
ITTM-random and ITTM-decidable random

Definition (Hamkins, Lewis)
A class of real $\mathcal{A}$ is **semi-decidable** if there is an ITTM $M$ such that $M(X) \downarrow$ if $X \in \mathcal{A}$.

Definition (Carl, Schlicht)
A sequence $X$ is **ITTM-random** if $X$ is in no semi-decidable set of measure 0.

Definition (Carl, Schlicht)
A sequence $X$ is **ITTM-decidable random** iff $X$ is in no decidable set of measure 0.
Lowness for $\lambda, \zeta, \Sigma$

**Definition**

We say that $X$ is low for $\lambda$ if $\lambda^X = \lambda$.

We say that $X$ is low for $\zeta$ if $\zeta^X = \zeta$.

We say that $X$ is low for $\Sigma$ if $\Sigma^X = \Sigma$.

**Theorem**

For any ordinal $\alpha$ with $\lambda \leq \alpha < \zeta$ we have $\lambda^\alpha > \lambda$ but:

1. $\zeta^\alpha = \zeta$.
2. $\Sigma^\alpha = \Sigma$.

(1) Indeed, suppose $\zeta$ is eventually writable using $\alpha$ and the machine $M$. As $\alpha$ is also eventually writable, we can run $M$ on every version of $\alpha$ and eventually write $\zeta$ which is a contradiction.

(2) Same argument.
Lowness for $\lambda, \zeta, \Sigma$

**Theorem**

The following are equivalent:

1. $\zeta^X > \zeta$.
2. $\Sigma^X > \Sigma$.
3. $\lambda^X > \Sigma$.

$(1) \rightarrow (2)$: We can again use the function:

$$f(0) = \zeta$$

$$f(n) = \text{the smallest } \alpha \text{ s.t. } C(\alpha) \upharpoonright n = C(\zeta) \upharpoonright n$$

To show that every ordinal $f(n)$ becomes eventually writable uniformly in $n$. Thus $\Sigma = \sup_n f(n)$ is also eventually writable.

$(2) \rightarrow (3)$: Define the machine that looks for the first pair of ordinals $\alpha < \beta$ such that $L_\alpha \prec_2 L_\beta$. Then write $\beta$. These ordinals must be $\zeta$ and $\Sigma$. 
Lowness for $\lambda$, $\zeta$, $\Sigma$ and randomness

**Theorem**

For any $X$ the triplet $(\lambda^X, \zeta^X, \Sigma^X)$ is the lexicographically least pair such that $L_{\lambda^X}[X] <_1 L_{\zeta^X}[X] <_2 L_{\Sigma^X}[X]$.

**Theorem (Carl, Schlicht)**

If $X$ is $(\Sigma + 1)$-random, then $L_{\lambda}[X] <_1 L_{\zeta}[X] <_2 L_{\Sigma}[X]$. In particular $\Sigma^X = \Sigma$, $\zeta^X = \zeta$ and $\lambda^X = \lambda$.

**Corollary (Carl, Schlicht)**

The set $\{X : \Sigma^X > \Sigma\}$ and $\{X : \lambda^X > \lambda\}$ are included in Borel sets of measure 0.
The ITTM model ITTM and constructibility The clockable ordinals ITTM and randomness

**ITTM-decidable randomness**

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**Theorem (Carl, Schlicht)**

The following are equivalent for a sequence $X$:

1. $X$ is ITTM-decidable random
2. $X$ is $\lambda$-random

Suppose some machine $M$ decides a set of measure 0 that $X$ belongs to. In particular it decides a set of measure 1 $X$ does not belong to. We have:

$$\lambda(\{X : M(X) \downarrow = 0\}) = 1$$

We then have

$$\lambda(\{X : M(X) \downarrow [\lambda] = 0\}) = 1$$

as the set of $X$ s.t. $\lambda^X = \lambda$ has measure 1. But then by admissibility:

$$\lambda(\{X : M(X) \downarrow [\alpha] = 0\}) = 1$$

already for some writable $\alpha$. The complement of this set is a Borel set of measure 0, with a writable code, and containing $X$. 
**Theorem (Carl, Schlicht)**

The following are equivalent for a sequence \( X \):

1. \( X \) is ITTM-random
2. \( X \) is \( \Sigma \)-random and \( \Sigma^X = \Sigma \)
3. \( X \) is \( \zeta \)-random and \( \Sigma^X = \Sigma \)

**Lemma (Carl, Schlicht)**

If \( \Sigma^X > \Sigma \), then \( X \) is not ITTM-random.

The set \( \{ X : \Sigma^X > \Sigma \} \) is an ITTM-semi-decidable set of measure 0. We saw that it is of measure 0. To see that it is ITTM-decidable, one can designe the machine which halts whenever it founds two \( X \)-accidentally writable ordinals \( \alpha < \beta \) such that \( L_\alpha <_2 L_\beta \).
Lemma (Carl, Schlicht)

If $X$ is not $\Sigma$-random, then $X$ is not ITTM-random.

If $X$ is not $\Sigma$-random, then with $X$ as an oracle, we can look for the first accidentally writable code for a Borel set of measure 0 containing $X$.

Lemma (Carl, Schlicht)

If $X$ is $\zeta$-random, but not ITTM-random, then $\Sigma^X > \Sigma$.

Suppose there is an ITTM $M$ which semi-decide a set of measure 0 containing $X$. Suppose $M(X) \downarrow [\alpha]$. Then we must have $\alpha \geq \zeta$ as otherwise the set $\{X : M(X) \downarrow [\alpha]\}$ would be a set of measure 0 with a Borel code in $L_{\zeta}$. Thus we must have $\lambda^X > \zeta$ and then $\Sigma^X > \Sigma$. 
ITTM-randomness

Question
Does there exist $X$ such that $X$ is $\Sigma$-random but not ITTM random?

Question
If $X$ is $\Sigma$-random, do we have $L_\zeta[X] <_2 L_\Sigma[X]$?