# Array noncomputability for left-c.e. reals and not totally $\omega$-c.e. degrees

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**Definition (Downey, Jockusch and Stob; Downey and Hirschfeldt)**

A sequence $\mathcal{F} = \{F_n\}_{n \geq 0}$ of finite sets is a *very strong array (v.s.a.)* if

(i) there is a computable function $f$ such that $f(n)$ is the canonical index of $F_n$,

(ii) $F_m \cap F_n = \emptyset$ if $m \neq n$, and

(iii) $0 < |F_n| < |F_{n+1}|$ for all $n \geq 0$.

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## Definition

Let $\mathcal{F} = \{F_n\}_{n \geq 0}$ be a v.s.a. and let $A$ and $B$ be any sets. $A$ and $B$ are *$\mathcal{F}$-similar* ($A \sim_{\mathcal{F}} B$ for short) if

$$\exists^{\infty} n \, (A \cap F_n = B \cap F_n).$$

**Definition (Downey, Jockusch and Stob)**

- Let $\mathcal{F} = \{F_n\}_{n \geq 0}$ be a v.s.a. A set $A$ is *$\mathcal{F}$-array noncomputable* (*$\mathcal{F}$-a.n.c.* for short) if $A$ is c.e. and, for any c.e. set $B$, $A$ and $B$ are $\mathcal{F}$-similar.
- A set $A$ is *array noncomputable* (*a.n.c.* for short) if $A$ is $\mathcal{F}$-a.n.c. for some v.s.a. $\mathcal{F}$.
- A c.e. degree **a** is *array noncomputable* (*a.n.c.* for short) if there is an a.n.c. set $A$ in **a**; and **a** is *array computable* otherwise.

The requirement to make a set a.n.c. may be weakened as follows.

**Proposition (Downey, Jockusch and Stob)**

*Let $\mathcal{F} = \{F_n\}_{n \geq 0}$ be a v.s.a. and let $A$ be a c.e. set such that, for any c.e. set $B$, the following holds.*

$$\exists n \, (A \cap F_n = B \cap F_n).$$

*Then $A$ is $\mathcal{F}$-a.n.c.*

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*Then $A$ is $\mathcal{F}$-a.n.c.*

Array noncomputability of a wtt-degree does not depend on the v.s.a. chosen and is closed upwards.

**Proposition (Downey, Jockusch and Stob)**

*Given very strong arrays $\mathcal{F}$ and $\mathcal{F}'$ and c.e. sets $A$ and $\hat{B}$ such that $A$ is $\mathcal{F}$-a.n.c. and $A \leq_{\text{wtt}} \hat{B}$, there is an $\mathcal{F}'$-a.n.c. set $B$ such that $B =_{\text{wtt}} \hat{B}$.*

## Definition

A function $f$ is *h-c.e.* for a function $h$ if there is a computable approximation $\{f_s\}_{s\geq 0}$ to $f$ such that the following holds for all $x$.

$$|\{s : f_{s+1}(x) \neq f_s(x)\}| \leq h(x).$$

## Lemma (Downey, Jockusch and Stob)

*The following are equivalent for a degree* $\mathbf{a}$.

(i) $\mathbf{a}$ *is a.n.c.*

(ii) *For every computable function* $h$*, there is a function* $f \leq_T \mathbf{a}$ *that is not* $h$*-c.e.*

(iii) *For any function* $g \leq_{wtt} \emptyset'$*, there is a function* $f \leq_T \mathbf{a}$ *that is not dominated by* $g$*.*

**Definition**

A degree $\mathbf{a}$ is *non-low$_2$* if for any function $g \leq_T \emptyset'$, there is a function $f \leq_T \mathbf{a}$ that is not dominated by $g$.

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It follows directly that any non-low$_2$ degree is a.n.c. However, it has been shown that there are low degrees which are a.n.c. As we have just seen, the a.n.c. wtt-degrees are closed upwards.

**Definition (Downey, Greenberg and Weber)**

A degree **a** is *totally* $\omega$*-c.e.* if for any function $f \leq_T \mathbf{a}$, there is a computable function $h$ such that $f$ is $h$-c.e.

In the following, we are interested in the *not* totally $\omega$-c.e. Turing degrees. It follows from the definition that those are closed upwards. Furthermore, the not totally $\omega$-c.e. Turing degrees are properly contained in the a.n.c. Turing degrees. It has been shown by Downey, Greenberg and Weber that the not totally $\omega$-c.e. Turing degrees are definable (they bound a critical triple).

**Theorem (Barmpalias, Downey and Greenberg)**

*A degree* $\mathbf{a}$ *is a.n.c. if and only if there is a left-c.e. set* $A \in \mathbf{a}$ *that is not cl-reducible to any random left-c.e. set.*

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**Theorem (Ambos-Spies, Losert and Monath)**

*A degree $\mathbf{a}$ is not totally $\omega$-c.e. if and only if there is a left-c.e. set $A \in \mathbf{a}$ that is not cl-reducible to any complex left-c.e. set.*

When studying properties of left-c.e. sets within the a.n.c. degrees, it is convenient to consider array noncomputability for left-c.e. sets. This cannot be done in the "most obvious" way, as for any v.s.a. $\mathcal{F}$, there is no left-c.e. set which is $\mathcal{F}$-similar to *all* left-c.e. sets.

However, we may consider sets that are "locally" left-c.e. with respect to a given v.s.a. $\mathcal{F}$. Then, there are indeed left-c.e. set which are $\mathcal{F}$-similar to any such set.

We will see that the Turing degrees of such sets coincide with the a.n.c. degrees.

## Definition

Let $\mathcal{F} = \{F_n\}_{n\geq 0}$ be a very strong array. A computable approximation $\{A_s\}_{s\geq 0}$ of $A$ is $\mathcal{F}$-*compatible* if, for any $n, s \geq 0$,

$$A_s \cap F_n \leq_{lex} A_{s+1} \cap F_n$$

and, for any $x \notin \bigcup_{n\geq 0} F_n$ and any $s \geq 0$, $A_s(x) \leq A_{s+1}(x)$.

A set $A$ is $\mathcal{F}$-*compatibly left-c.e.* ($\mathcal{F}$-*left-c.e.* or $\mathcal{F}$-*l.c.e.*) if there is an $\mathcal{F}$-compatible approximation $\{A_s\}_{s\geq 0}$ of $A$.

## Definition

- Let $\mathcal{F} = \{F_n\}_{n \geq 0}$ be a very strong array. A set $A$ is *$\mathcal{F}$-array noncomputable for the $\mathcal{F}$-l.c.e. sets* (*$\mathcal{F}$-l.c.e.-a.n.c.*) if $A$ is l.c.e. and, for all $\mathcal{F}$-l.c.e. sets $B$, $A \sim_{\mathcal{F}} B$.
- A set $A$ is *array noncomputable for the left-c.e. sets* (*l.c.e.-a.n.c.*) if $A$ is $\mathcal{F}$-l.c.e.-a.n.c. for some v.s.a. $\mathcal{F}$.
- A degree **a** is *l.c.e.-a.n.c.* if it contains an l.c.e.-a.n.c. set.

**Theorem**

*Let $\mathcal{F} = \{F_n\}_{n \geq 0}$ be a v.s.a. Then the following hold.*

- *For any l.c.e.-a.n.c. set $A$ there is an $\mathcal{F}$-a.n.c. set $B$ such that $A =_{\text{wtt}} B$.*
- *For any a.n.c. set $A$ there is an $\mathcal{F}$-l.c.e.-a.n.c. set $B$ such that $A =_{\text{wtt}} B$.*

As the a.n.c. wtt-degrees are closed upwards, this implies that the same holds for the l.c.e.-a.n.c. wtt-degrees.

As shown by Downey, Jockusch and Stob, no c.e. set can be $\mathcal{F}$-a.n.c. for *every* v.s.a. $\mathcal{F}$. For the case of l.c.e.-array noncomputability, however, such universal sets do exist.

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## Definition

- An l.c.e. set $A$ is *universally l.c.e.-a.n.c.* if $A$ is $\mathcal{F}$-l.c.e.-a.n.c. for all very strong arrays $\mathcal{F}$, i.e., if, for any v.s.a. $\mathcal{F}$ and any $\mathcal{F}$-l.c.e. set $B$, $A$ is $\mathcal{F}$-similar to $B$.
- A degree is *universally l.c.e.-a.n.c.* if it contains a universally l.c.e.-a.n.c. set.

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- A degree is *universally l.c.e.-a.n.c.* if it contains a universally l.c.e.-a.n.c. set.

It turns out that the universally l.c.e.-a.n.c. Turing degrees coincide with the not totally $\omega$-c.e. Turing degrees.

## Theorem

*A Turing degree* $\mathbf{a}$ *is not totally* $\omega$*-c.e. if and only if it is universally l.c.e.-a.n.c.*

Recall that the a.n.c. wtt-degrees are closed upwards. Furthermore, by coincidence with the not totally $\omega$-c.e. degrees, the universally l.c.e.-a.n.c. Turing degrees are closed upwards, too. This might lead one to conjecture that the universally l.c.e.-a.n.c. *wtt*-degrees are closed upwards as well. However, this is not the case. In fact, we have the following.

**Theorem**

*No wtt-hard set is universally l.c.e.-a.n.c.*

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**Theorem**

*No wtt-hard set is universally l.c.e.-a.n.c.*

**Definition (Kanovich)**

Let $h$ be a computable order. A set $A$ is *$h$-complex* if $C(A \upharpoonright n) \geq h(n)$ for all $n$. A set $A$ is *complex* if $A$ is $h$-complex for some computable order.

By a result of Kanovich, we may replace wtt-hard with complex.

**Lemma**

*Let $A$ be universally l.c.e.-a.n.c. Then, $A$ is not complex.*

By definition of universally l.c.e.-a.n.c. sets, it is enough to prove the following.

**Lemma**

*Let $h$ be a computable order. There is a v.s.a. $\mathcal{F} = \{F_n\}_{n \geq 0}$ such that any set $A$ which is $\mathcal{F}$-similar to the empty set is not $h$-complex.*

## Definition

- For any function $f : \omega \to \omega$, an ML-test $\{U_n\}_{n\geq 0}$ is *$f$-bounded* (an *$f$-test*) if, for $n \geq 0$, $|U_n| \leq f(n)$.
- A set $A$ is *$f$-Martin-Löf random* if $A$ passes all $f$-tests.
- A set $A$ is *computably-bounded random* (*CB-random*) if $A$ is $f$-ML-random for all computable functions $f$.

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## Theorem (Downey, Brodhead, Ng)

*Let* **a** *be a not totally $\omega$-c.e. Turing degree. Then,* **a** *contains a set which is CB-random. Furthermore, there is a left-c.e. set $A \leq_T$* **a** *which is CB-random.*

As we have seen,universally l.c.e.-a.n.c. sets are not complex, so they are not ML-random, either. However, they are CB-random.

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*Any universally l.c.e.-a.n.c. set is CB-random.*

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**Theorem**

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It is enough to show the following.

**Lemma**

*Let $f$ be a computable function. There is a v.s.a. $\mathcal{F} = \{F_n\}_{n \geq 0}$ such that any $\mathcal{F}$-l.c.e.-a.n.c. set $A$ passes any $f$-ML-test.*

Note that $\mathcal{F}$ only depends on the function $f$ but not on the particular $f$-bounded ML-test.

**Theorem (Ambos-Spies, Losert and Monath)**

*If* $\mathbf{a}$ *is not totally* $\omega$*-c.e. then there is a left-c.e. set* $A \in \mathbf{a}$ *that is not cl-reducible to any complex left-c.e. set.*

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Again by Kanovich's result, we may replace complex with wtt-hard. Moreover, we may replace cl-reducible with ibT-reducible in this context. By our result on universally l.c.e.-a.n.c. degrees, the theorem follows directly from the following lemma.

**Lemma**

*If* $A$ *is universally l.c.e.-a.n.c. then* $A$ *is not ibT-reducible to any wtt-hard left-c.e. set.*

By the following equivalence, this formulation of the theorem is tightly related to maximal pairs in the l.c.e. ibT-degrees.

**Lemma**

*Let $A$ be a left-c.e. set. The following are equivalent.*

(i) *$A$ is not ibT-reducible to any wtt-hard left-c.e. set.*

(ii) *For any infinite computable set $D$ there is a computably enumerable subset $B$ of $D$ such that $(A, B)$ is an ibT-maximal pair in the left-c.e. sets.*

Yu and Ding have shown that there exists a maximal pair in the left-c.e. ibT-degrees. This result has been extended in various directions. E.g., Fan has shown that there is a maximal pair in the left-c.e. ibT-degrees such that one half is c.e. In fact, by a result of Fan and Yu, *every* left-c.e. set is half of a maximal pair. However, as shown by Downey and Hirschfeldt, we cannot make both halves c.e.

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In order to get our result it suffices to prove the following lemma extending Fan's result in two directions.

**Lemma**

*Let $A$ be a universally l.c.e.-a.n.c. set and let $D$ be any infinite computable set. There is a c.e. set $B \subseteq D$ such that $(A, B)$ is an ibT-maximal pair in the left-c.e. sets.*

By analyzing and slightly changing Fan's construction, we obtain the following which implies the above lemma.

### Lemma

*Let $D$ be an infinite computable set. There is a computable function $l$ such that the following hold. For any ibT-functionals $\hat{\Phi}$ and $\hat{\Psi}$, any left-c.e. set $V$ and any number $a \geq 0$, there are uniformly (in $\hat{\Phi}$, $\hat{\Psi}$, $V$ and $a$) left-c.e. reals*

$$A_a^{\hat{\Phi},\hat{\Psi},V} \subseteq [a, a + l(a)]$$

*and uniformly (in $\hat{\Phi}$, $\hat{\Psi}$, $V$ and $a$) c.e. sets*

$$B_a^{\hat{\Phi},\hat{\Psi},V} \subseteq [a, a + l(a)] \cap D$$

*such that the following holds.*

$$\exists\, x \in [a, a + l(a)]\ (A_a^{\hat{\Phi},\hat{\Psi},V}(x) \neq \hat{\Phi}^V(x) \text{ or } B_a^{\hat{\Phi},\hat{\Psi},V}(x) \neq \hat{\Psi}^V(x)).$$

# Thank you!