Array noncomputability for left-c.e. reals and not totally $\omega$-c.e. degrees

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A sequence $\mathcal{F} = \{F_n\}_{n\geq 0}$ of finite sets is a **very strong array** (v.s.a.) if

(i) there is a computable function $f$ such that $f(n)$ is the canonical index of $F_n$,

(ii) $F_m \cap F_n = \emptyset$ if $m \neq n$, and

(iii) $0 < |F_n| < |F_{n+1}|$ for all $n \geq 0$. 
Definition (Downey, Jockusch and Stob; Downey and Hirschfeldt)

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Definition

Let $\mathcal{F} = \{F_n\}_{n \geq 0}$ be a v.s.a. and let $A$ and $B$ be any sets. $A$ and $B$ are $\mathcal{F}$-similar ($A \sim \mathcal{F} B$ for short) if

$$\exists \infty n \ (A \cap F_n = B \cap F_n).$$
Definition (Downey, Jockusch and Stob)

- Let $\mathcal{F} = \{F_n\}_{n \geq 0}$ be a v.s.a. A set $A$ is $\mathcal{F}$-array noncomputable ($\mathcal{F}$-a.n.c. for short) if $A$ is c.e. and, for any c.e. set $B$, $A$ and $B$ are $\mathcal{F}$-similar.

- A set $A$ is array noncomputable (a.n.c. for short) if $A$ is $\mathcal{F}$-a.n.c. for some v.s.a. $\mathcal{F}$.

- A c.e. degree $a$ is array noncomputable (a.n.c. for short) if there is an a.n.c. set $A$ in $a$; and $a$ is array computable otherwise.
The requirement to make a set a.n.c. may be weakened as follows.

**Proposition (Downey, Jockusch and Stob)**

Let \( \mathcal{F} = \{ F_n \}_{n \geq 0} \) be a v.s.a. and let \( A \) be a c.e. set such that, for any c.e. set \( B \), the following holds.

\[
\exists n \ ( A \cap F_n = B \cap F_n ).
\]

Then \( A \) is \( \mathcal{F} \)-a.n.c.
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**Proposition (Downey, Jockusch and Stob)**

Let $\mathcal{F} = \{F_n\}_{n \geq 0}$ be a v.s.a. and let $A$ be a c.e. set such that, for any c.e. set $B$, the following holds.

$$\exists n (A \cap F_n = B \cap F_n).$$

Then $A$ is $\mathcal{F}$-a.n.c.

Array noncomputability of a wtt-degree does not depend on the v.s.a. chosen and is closed upwards.

**Proposition (Downey, Jockusch and Stob)**

Given very strong arrays $\mathcal{F}$ and $\mathcal{F}'$ and c.e. sets $A$ and $\hat{B}$ such that $A$ is $\mathcal{F}$-a.n.c. and $A \leq_{wtt} \hat{B}$, there is an $\mathcal{F}'$-a.n.c. set $B$ such that $B =_{wtt} \hat{B}$. 
Definition

A function $f$ is $h$-c.e. for a function $h$ if there is a computable approximation $\{f_s\}_{s \geq 0}$ to $f$ such that the following holds for all $x$.

$$|\{s : f_{s+1}(x) \neq f_s(x)\}| \leq h(x).$$

Lemma (Downey, Jockusch and Stob)

The following are equivalent for a degree $a$.

(i) $a$ is a.n.c.

(ii) For every computable function $h$, there is a function $f \leq_T a$ that is not $h$-c.e.

(iii) For any function $g \leq_{wtt} \emptyset'$, there is a function $f \leq_T a$ that is not dominated by $g$. 
Definition

A degree $a$ is non-$\text{low}_2$ if for any function $g \leq_T \emptyset'$, there is a function $f \leq_T a$ that is not dominated by $g$. 
A degree \( a \) is *non-low* \(_2\) if for any function \( g \leq_T \emptyset' \), there is a function \( f \leq_T a \) that is not dominated by \( g \).

It follows directly that any non-low \(_2\) degree is a.n.c. However, it has been shown that there are low degrees which are a.n.c. As we have just seen, the a.n.c. wtt-degrees are closed upwards.
Definition (Downey, Greenberg and Weber)

A degree \( a \) is *totally* \( \omega \)-c.e. if for any function \( f \leq_T a \), there is a computable function \( h \) such that \( f \) is \( h \)-c.e.

In the following, we are interested in the *not* totally \( \omega \)-c.e. Turing degrees. It follows from the definition that those are closed upwards. Furthermore, the not totally \( \omega \)-c.e. Turing degrees are properly contained in the a.n.c. Turing degrees. It has been shown by Downey, Greenberg and Weber that the not totally \( \omega \)-c.e. Turing degrees are definable (they bound a critical triple).
Theorem (Barmpalias, Downey and Greenberg)

A degree $a$ is a.n.c. if and only if there is a left-c.e. set $A \in a$ that is not cl-reducible to any random left-c.e. set.
Theorem (Barmpalias, Downey and Greenberg)

A degree $a$ is a.n.c. if and only if there is a left-c.e. set $A \in a$ that is not cl-reducible to any random left-c.e. set.

Theorem (Ambos-Spies, Losert and Monath)

A degree $a$ is not totally $\omega$-c.e. if and only if there is a left-c.e. set $A \in a$ that is not cl-reducible to any complex left-c.e. set.
When studying properties of left-c.e. sets within the a.n.c. degrees, it is convenient to consider array noncomputability for left-c.e. sets. This cannot be done in the “most obvious” way, as for any v.s.a. $\mathcal{F}$, there is no left-c.e. set which is $\mathcal{F}$-similar to all left-c.e. sets.

However, we may consider sets that are “locally” left-c.e. with respect to a given v.s.a. $\mathcal{F}$. Then, there are indeed left-c.e. set which are $\mathcal{F}$-similar to any such set.

We will see that the Turing degrees of such sets coincide with the a.n.c. degrees.
Definition

Let $\mathcal{F} = \{F_n\}_{n \geq 0}$ be a very strong array. A computable approximation $\{A_s\}_{s \geq 0}$ of $A$ is $\mathcal{F}$-compatible if, for any $n, s \geq 0$,

$$A_s \cap F_n \leq_{\text{lex}} A_{s+1} \cap F_n$$

and, for any $x \notin \bigcup_{n \geq 0} F_n$ and any $s \geq 0$, $A_s(x) \leq A_{s+1}(x)$.

A set $A$ is $\mathcal{F}$-compatibly left-c.e. ($\mathcal{F}$-left-c.e. or $\mathcal{F}$-l.c.e.) if there is an $\mathcal{F}$-compatible approximation $\{A_s\}_{s \geq 0}$ of $A$. 
Definition

- Let $\mathcal{F} = \{F_n\}_{n \geq 0}$ be a very strong array. A set $A$ is $\mathcal{F}$-array noncomputable for the $\mathcal{F}$-l.c.e. sets ($\mathcal{F}$-l.c.e.-a.n.c.) if $A$ is l.c.e. and, for all $\mathcal{F}$-l.c.e. sets $B$, $A \sim_{\mathcal{F}} B$.

- A set $A$ is array noncomputable for the left-c.e. sets (l.c.e.-a.n.c.) if $A$ is $\mathcal{F}$-l.c.e.-a.n.c. for some v.s.a. $\mathcal{F}$.

- A degree $\mathbf{a}$ is l.c.e.-a.n.c. if it contains an l.c.e.-a.n.c. set.
Theorem

Let $\mathcal{F} = \{ F_n \}_{n \geq 0}$ be a v.s.a. Then the following hold.

- For any l.c.e.-a.n.c. set $A$ there is an $\mathcal{F}$-a.n.c. set $B$ such that $A =_{\text{wtt}} B$.
- For any a.n.c. set $A$ there is an $\mathcal{F}$-l.c.e.-a.n.c. set $B$ such that $A =_{\text{wtt}} B$.

As the a.n.c. wtt-degrees are closed upwards, this implies that the same holds for the l.c.e.-a.n.c. wtt-degrees.
As shown by Downey, Jockusch and Stob, no c.e. set can be $\mathcal{F}$-a.n.c. for every v.s.a. $\mathcal{F}$. For the case of l.c.e.-array noncomputability, however, such universal sets do exist.
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**Definition**

- An l.c.e. set $A$ is *universally l.c.e.-a.n.c.* if $A$ is $\mathcal{F}$-l.c.e.-a.n.c. for all very strong arrays $\mathcal{F}$, i.e., if, for any v.s.a. $\mathcal{F}$ and any $\mathcal{F}$-l.c.e. set $B$, $A$ is $\mathcal{F}$-similar to $B$.
- A degree is *universally l.c.e.-a.n.c.* if it contains a universally l.c.e.-a.n.c. set.
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- A degree is *universally l.c.e.-a.n.c.* if it contains a universally l.c.e.-a.n.c. set.

It turns out that the universally l.c.e.-a.n.c. Turing degrees coincide with the not totally $\omega$-c.e. Turing degrees.

### Theorem

A Turing degree $\mathbf{a}$ is not totally $\omega$-c.e. if and only if it is universally l.c.e.-a.n.c.
Recall that the a.n.c. wtt-degrees are closed upwards. Furthermore, by coincidence with the not totally $\omega$-c.e. degrees, the universally l.c.e.-a.n.c. Turing degrees are closed upwards, too. This might lead one to conjecture that the universally l.c.e.-a.n.c. wtt-degrees are closed upwards as well. However, this is not the case. In fact, we have the following.

**Theorem**

No wtt-hard set is universally l.c.e.-a.n.c.
Recall that the a.n.c. wtt-degrees are closed upwards. Furthermore, by coincidence with the not totally \( \omega \)-c.e. degrees, the universally l.c.e.-a.n.c. Turing degrees are closed upwards, too. This might lead one to conjecture that the universally l.c.e.-a.n.c. wtt-degrees are closed upwards as well. However, this is not the case. In fact, we have the following.

**Theorem**

*No wtt-hard set is universally l.c.e.-a.n.c.*

**Definition (Kanovich)**

Let \( h \) be a computable order. A set \( A \) is *\( h \)-complex* if \( C(A \upharpoonright n) \geq h(n) \) for all \( n \). A set \( A \) is *complex* if \( A \) is \( h \)-complex for some computable order.
By a result of Kanovich, we may replace wtt-hard with complex.

**Lemma**

*Let $A$ be universally l.c.e.-a.n.c. Then, $A$ is not complex.*

By definition of universally l.c.e.-a.n.c. sets, it is enough to prove the following.

**Lemma**

*Let $h$ be a computable order. There is a v.s.a. $\mathcal{F} = \{F_n\}_{n \geq 0}$ such that any set $A$ which is $\mathcal{F}$-similar to the empty set is not $h$-complex.*
## Definition

- For any function $f : \omega \to \omega$, an ML-test $\{U_n\}_{n \geq 0}$ is **$f$-bounded** (an $f$-test) if, for $n \geq 0$, $|U_n| \leq f(n)$.

- A set $A$ is **$f$-Martin-Löf random** if $A$ passes all $f$-tests.

- A set $A$ is **computably-bounded random** (CB-random) if $A$ is $f$-ML-random for all computable functions $f$. 

Theorem (Downey, Brodhead, Ng)

Let $a$ be a not totally $\omega$-c.e. Turing degree. Then, $\exists a$ contains a set which is CB-random. Furthermore, there is a left-c.e. set $A \leq_T a$ which is CB-random.
**Definition**

- For any function $f : \omega \rightarrow \omega$, an ML-test $\{U_n\}_{n \geq 0}$ is $f$-bounded (an $f$-test) if, for $n \geq 0$, $|U_n| \leq f(n)$.
- A set $A$ is $f$-Martin-Löf random if $A$ passes all $f$-tests.
- A set $A$ is computably-bounded random (CB-random) if $A$ is $f$-ML-random for all computable functions $f$.

**Theorem (Downey, Brodhead, Ng)**

*Let $a$ be a not totally $\omega$-c.e. Turing degree. Then, $a$ contains a set which is CB-random. Furthermore, there is a left-c.e. set $A \leq_T a$ which is CB-random.*
As we have seen, universally l.c.e.-a.n.c. sets are not complex, so they are not ML-random, either. However, they are CB-random.

**Theorem**

*Any universally l.c.e.-a.n.c. set is CB-random.*
As we have seen, universally l.c.e.-a.n.c. sets are not complex, so they are not ML-random, either. However, they are CB-random.

**Theorem**

*Any universally l.c.e.-a.n.c. set is CB-random.*

It is enough to show the following.

**Lemma**

*Let $f$ be a computable function. There is a v.s.a. $\mathcal{F} = \{F_n\}_{n \geq 0}$ such that any $\mathcal{F}$-l.c.e.-a.n.c. set $A$ passes any $f$-ML-test.*

Note that $\mathcal{F}$ only depends on the function $f$ but not on the particular $f$-bounded ML-test.
Theorem (Ambos-Spies, Losert and Monath)

If \( a \) is not totally \( \omega \)-c.e. then there is a left-c.e. set \( A \in a \) that is not cl-reducible to any complex left-c.e. set.
Theorem (Ambos-Spies, Losert and Monath)

*If \( a \) is not totally \( \omega \)-c.e. then there is a left-c.e. set \( A \in a \) that is not cl-reducible to any complex left-c.e. set.*

Again by Kanovich’s result, we may replace complex with wtt-hard. Moreover, we may replace cl-reducible with ibT-reducible in this context. By our result on universally l.c.e.-a.n.c. degrees, the theorem follows directly from the following lemma.

Lemma

*If \( A \) is universally l.c.e.-a.n.c. then \( A \) is not ibT-reducible to any wtt-hard left-c.e. set.*
By the following equivalence, this formulation of the theorem is tightly related to maximal pairs in the l.c.e. ibT-degrees.

**Lemma**

Let A be a left-c.e. set. The following are equivalent.

(i) A is not ibT-reducible to any wtt-hard left-c.e. set.

(ii) For any infinite computable set D there is a computably enumerable subset B of D such that (A, B) is an ibT-maximal pair in the left-c.e. sets.
Yu and Ding have shown that there exists a maximal pair in the left-c.e. ibT-degrees. This result has been extended in various directions. E.g., Fan has shown that there is a maximal pair in the left-c.e. ibT-degrees such that one half is c.e. In fact, by a result of Fan and Yu, every left-c.e. set is half of a maximal pair. However, as shown by Downey and Hirschfeldt, we cannot make both halves c.e.
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In order to get our result it suffices to prove the following lemma extending Fan’s result in two directions.

**Lemma**

Let $A$ be a universally l.c.e.-a.n.c. set and let $D$ be any infinite computable set. There is a c.e. set $B \subseteq D$ such that $(A, B)$ is an ibT-maximal pair in the left-c.e. sets.
By analyzing and slightly changing Fan’s construction, we obtain the following which implies the above lemma.

**Lemma**

Let $D$ be an infinite computable set. There is a computable function $l$ such that the following hold. For any ibT-functionals $\hat{\Phi}$ and $\hat{\Psi}$, any left-c.e. set $V$ and any number $a \geq 0$, there are uniformly (in $\hat{\Phi}$, $\hat{\Psi}$, $V$ and $a$) left-c.e. reals $A_{a,\hat{\Phi},\hat{\Psi},V} \subseteq [a, a + l(a)]$ and uniformly (in $\hat{\Phi}$, $\hat{\Psi}$, $V$ and $a$) c.e. sets $B_{a,\hat{\Phi},\hat{\Psi},V} \subseteq [a, a + l(a)] \cap D$ such that the following holds.

$$\exists x \in [a, a + l(a)] (A_{a,\hat{\Phi},\hat{\Psi},V}(x) \neq \hat{\Phi}^V(x) \text{ or } B_{a,\hat{\Phi},\hat{\Psi},V}(x) \neq \hat{\Psi}^V(x)).$$
Thank you!