Increasing dimension s to dimension t with few changes

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Randomness and effective dimension 1

Observation: You can make sequences of effective dimension 1 by flipping density zero bits on a random.

Question 1 (Rod): Can you make every sequence of effective dimension 1 that way?

Yes!

Theorem 1: The sequences of effective dimension 1 are exactly the sequences which differ on a density zero set from a ML random sequence.

Decreasing from dimension 1 to dimension $s < 1$

Observation: You can make sequences of effective dimension 1/2 by changing all odd bits of a random to 0. Density of changes: 1/4.

Question 2: Can we change a random on fewer than 1/4 of the bits and still make a sequence of effective dimension 1/2?

Decreasing from dimension 1 to dimension $s < 1$

A naive bound on the distance needed:
Proposition: If $\overline{\rho}(X \Delta Y) = d$, then

$$\dim X \leq \dim Y + H(d)$$

where H is Shannon's binary entropy function $H(p) = -(p \log p + (1-p)\log(1-p))$.

So if $\dim X = 1$ and we want to find nearby Y with $\dim Y = s$, then we will need to use distance at least $d = H^{-1}(1-s)$.

Yes! (to Question 2)
Theorem 2: For any X with $\dim X = 1$ and any $s < 1$, there is Y with $d(X, Y) = H^{-1}(1-s)$ and $\dim(Y) = s$.

where $d(X, Y) = \overline{\rho}(X \Delta Y)$.

Notation

Write $X = \sigma_1\sigma_2\dots$ where $|\sigma_i| = i^2$.

Let $\dim(\sigma) = K(\sigma)/|\sigma|$.

Let $s_i = \dim(\sigma_i|\sigma_1\dots\sigma_{i-1})$

Fact:

$$\dim(\sigma_1\dots\sigma_i) \approx \sum_{k=1}^{i} \frac{|\sigma_k|}{|\sigma_1\dots\sigma_i|} s_k$$

Also:

$$\rho(\sigma_1\dots\sigma_i) = \sum_{k=1}^{k} \frac{|\sigma_k|}{|\sigma_1\dots\sigma_i|} \rho(\sigma_k),$$

where $\rho(\sigma) = (\#\text{ of 1s in } \sigma)/|\sigma|$.

Decreasing from dimension 1 to dimension s

Fact: For any σ and any $s < 1$, there is τ with $\rho(\sigma\Delta\tau) \leq H^{-1}(1-s)$ and $\dim(\tau) \leq s$.

(using basic Vereschagin-Vitanyi theory)

Theorem 2: For any X with $\dim X = 1$ and any $s < 1$, there is Y with $d(X, Y) = H^{-1}(1-s)$ and $\dim(Y) = s$.

Proof: Given $X = \sigma_1\sigma_2\ldots$, produce $Y = \tau_1\tau_2\ldots$, where τ_i is obtained from σ_i by applying the above fact.

Each $\dim(\tau_i) \leq s$ and each $\rho(\sigma_i\Delta\tau_i) \leq H^{-1}(1-s)$, so Y and $X\Delta Y$ satisfy these bounds in the limit. □

Increasing from dimension s to dimension 1

Observation: Consider a Bernoulli p-random X (obtained by flipping a coin with probability p of getting a 1). We have $\dim(X) = H(p)$ and $\rho(X) = p$.

Obviously, we will need at least density $1/2 - p$ of changes to bring the density up to $1/2$, a necessary pre-requisite for bringing the effective dimension to 1.

Proposition: For each s, there is X with $\dim(X) = s$ such that for all Y with $\dim(Y) = 1$, we have $\overline{\rho}(X \Delta Y) \geq 1/2 - H^{-1}(s)$.

(X is any Bernoulli $H^{-1}(s)$-random.)

Theorem 3: For any $s < 1$ and any X with $\dim(X) = s$, there is Y with $\dim(Y) = 1$ and $d(X, Y) \leq 1/2 - H^{-1}(s)$.

A finite increasing theorem

Fact: For any σ, s, t with $\dim(\sigma) = s < t \leq 1$, there is τ with $\rho(\sigma \Delta \tau) \leq H^{-1}(t) - H^{-1}(s)$ and $\dim(\tau) = t$.

(more basic Vereshchagin-Vitanyi theory)

The Main Lemma

Let $X = \sigma_1\sigma_2\dots$ where $|\sigma_i| = i^2$.

Recall $s_i = \dim(\sigma_i|\sigma_1\dots\sigma_{i-1})$.

Lemma: Let $t_1, t_2, \dots,$ and $d_1, d_2 \dots$ be any sequences satisfying for all i,

$$d_i = H^{-1}(t_i) - H^{-1}(s_i).$$

Then there is $Y = \tau_1\tau_2\dots$ such that for all i,

$$t_i \leq \dim(\tau_i|\tau_1\dots\tau_{i-1}) \qquad \text{and} \qquad \rho(\sigma_i\Delta\tau_i) \leq d_i.$$

Proof: Uses Harper's Theorem and compactness.

A convexity argument

Given $X = \sigma_1\sigma_2\ldots$ with $\dim(X) = s$, we want to produce $Y = \tau_1\tau_2\ldots$ with $\dim(Y) = 1$ and $d(X,Y) \leq 1/2 - H^{-1}(s)$.

Let $t_i = 1$ for all i. Let $d_i = 1/2 - H^{-1}(s_i)$. Let Y be as guaranteed by the Main Lemma. Then

$$\dim(Y) = \liminf_i \sum_{k=1}^{i} \frac{|\tau_k|}{|\tau_1\ldots\tau_i|} t_k = 1$$

$$d(X,Y) = \limsup_i \sum_{k=1}^{i} \frac{|\tau_k|}{|\tau_1\ldots\tau_i|}(1/2 - H^{-1}(s_i))$$

$$\leq 1/2 - H^{-1}(\liminf_i \sum_{k=1}^{i} \frac{|\tau_k|}{|\tau_1\ldots\tau_i|} s_i) = 1/2 - H^{-1}(s)$$

because $s_i \mapsto 1/2 - H^{-1}(s_i)$ is concave.

Summary of the Preparation

Increasing dimension s to dimension 1:

- Distance at least $1/2 - H^{-1}(s)$ may be needed to handle starting with a Bernoulli $H^{-1}(s)$-random.
- This distance suffices (construction).

Decreasing dimension 1 to dimension s:

- Distance at least $H^{-1}(1 - s)$ is needed for information coding reasons.
- This distance suffices (construction).

Generalization goal

Increasing dimension s to dimension t:

- Distance at least $H^{-1}(t) - H^{-1}(s)$ may be needed to handle starting with a Bernoulli $H^{-1}(s)$-random.
- Construction breaks (convexity)

Decreasing dimension t to dimension s:

- Distance at least $H^{-1}(t - s)$ is needed for information coding reasons.
- Construction breaks (even finite version)

Failure of convexity I (increasing from s to t)

Strategy: Pump all information density up to t.

Problem: setting all $t_i = t$ in the Main Lemma, the map $s_i \mapsto d_i = H^{-1}(t_i) - H^{-1}(s_i)$ is not concave.

(on the board)

Failures of convexity II (increasing from s to t)

Strategy: Constant distance. Let $d = H^{-1}(t) - H^{-1}(s)$, pump in as much information as possible within distance d.

Problem: setting all $d_i = d$ in the Main Lemma, the map $s_i \mapsto t_i = H(d_i + H^{-1}(s_i))$ is not convex (except at some small values of s_i).

(on the board)

Line toeing strategy

Theorem 3+: For any $s < t \leq 1$ and any X with $\dim(X) = s$, there is Y with $\dim(Y) = t$ and $d(X, Y) \leq H^{-1}(t) - H^{-1}(s)$.

Proof uses the following strategy:

Given s_i, set t_i so that (s_i, t_i) lies on the line connecting (s, t) and $(1, 1)$.

This produces a map $s_i \mapsto d_i$ which is concave!!

(on the board)

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(on the board)

(seven derivatives later, including a partial derivative with respect to one of the parameters, we prove this map is concave.)

Pairs (s, t) for which the line toeing strategy works

Problem: This map only works for pairs (s, t) such that the map $s_i \to d_i$ is decreasing at s.

After some undergraduate calculus, these are exactly the pairs (s, t) satisfying

$$(1-t)g'(t) \leq (1-s)g'(s)$$

where $g = H^{-1}$.

(on board)

We see that the line toeing strategy fails for some small values of s.

Constant distance strategy, reprise

We have already seen a strategy that only succeeds on some small values of s – the constant distance strategy.

(only four derivatives needed to show that $s_i \mapsto t_i$ has the required convexity properties for small s!)

Constant distance strategy, reprise

We have already seen a strategy that only succeeds on some small values of s – the constant distance strategy.

(only four derivatives needed to show that $s_i \mapsto t_i$ has the required convexity properties for small s!)

After some undergraduate calculus, the pairs (s, t) for which the constant distance strategy works are exactly those satisfying

$$(1-t)g'(t) \geq (1-s)g'(s)$$

where $g = H^{-1}$.

Yes, I really meant that

Line toeing strategy works at (s,t) if and only if

$$(1-t)g'(t) \leq (1-s)g'(s)$$

Constant distance strategy works at (s,t) if and only if

$$(1-t)g'(t) \geq (1-s)g'(s)$$

where $g = H^{-1}$.

For every $s < t \leq 1$, there is a working strategy (there is a way to set the t_i, d_i in the Main Lemma so that by convexity, the resulting Y has the right effective dimension and the right distance from a given X).

This proves Theorem 3+.

This is too precise to be a coincidence!?

Summary of the talk

Increasing dimension s to dimension t:

- Distance at least $H^{-1}(t) - H^{-1}(s)$ may be needed to handle starting with a Bernoulli $H^{-1}(s)$-random.
- This distance suffices (construction)

Decreasing dimension t to dimension s:

- Distance at least $H^{-1}(t-s)$ is needed for information coding reasons.
- Construction breaks (even finite version)
- In fact, this distance is demonstrably too short.

Questions

Given $s < t < 1$, what is the minimum distance d such that for every X with $\dim(X) = t$, there is a Y with $\dim(Y) = s$ and $d(X, Y) \leq d$?

Why do the line-toeing and constant-distance strategies dovetail so perfectly?