Increasing dimension $s$ to dimension $t$ with few changes

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Observation: You can make sequences of effective dimension 1 by flipping density zero bits on a random.

Question 1 (Rod): Can you make every sequence of effective dimension 1 that way?

Yes!

Theorem 1: The sequences of effective dimension 1 are exactly the sequences which differ on a density zero set from a ML random sequence.
Observation: You can make sequences of effective dimension $1/2$ by changing all odd bits of a random to 0. Density of changes: $1/4$.

Question 2: Can we change a random on fewer than $1/4$ of the bits and still make a sequence of effective dimension $1/2$?
A naive bound on the distance needed:

**Proposition:** If $\overline{\rho}(X \Delta Y) = d$, then

$$\dim X \leq \dim Y + H(d)$$

where $H$ is Shannon’s binary entropy function $H(p) = -(p \log p + (1 - p) \log(1 - p))$.

So if $\dim X = 1$ and we want to find nearby $Y$ with $\dim Y = s$, then we will need to use distance at least $d = H^{-1}(1 - s)$.

Yes! (to Question 2)

**Theorem 2:** For any $X$ with $\dim X = 1$ and any $s < 1$, there is $Y$ with $d(X, Y) = H^{-1}(1 - s)$ and $\dim(Y) = s$.

where $d(X, Y) = \overline{\rho}(X \Delta Y)$. 
Write $X = \sigma_1\sigma_2\ldots$ where $|\sigma_i| = i^2$.

Let $\dim(\sigma) = K(\sigma)/|\sigma|$.

Let $s_i = \dim(\sigma_i|\sigma_1\ldots\sigma_{i-1})$

**Fact:**

$$\dim(\sigma_1\ldots\sigma_i) \approx \sum_{k=1}^{i} \frac{|\sigma_k|}{|\sigma_1\ldots\sigma_i|} s_k$$

**Also:**

$$\rho(\sigma_1\ldots\sigma_i) = \sum_{k=1}^{k} \frac{|\sigma_k|}{|\sigma_1\ldots\sigma_i|} \rho(\sigma_k),$$

where $\rho(\sigma) = (# \text{ of } 1s \text{ in } \sigma)/|\sigma|$.
Fact: For any $\sigma$ and any $s < 1$, there is $\tau$ with $\rho(\sigma \Delta \tau) \leq H^{-1}(1 - s)$ and $\dim(\tau) \leq s$.

(using basic Vereschagin-Vitanyi theory)

Theorem 2: For any $X$ with $\dim X = 1$ and any $s < 1$, there is $Y$ with $d(X, Y) = H^{-1}(1 - s)$ and $\dim(Y) = s$.

Proof: Given $X = \sigma_1 \sigma_2 \ldots$, produce $Y = \tau_1 \tau_2 \ldots$, where $\tau_i$ is obtained from $\sigma_i$ by applying the above fact.

Each $\dim(\tau_i) \leq s$ and each $\rho(\sigma_i \Delta \tau_i) \leq H^{-1}(1 - s)$, so $Y$ and $X \Delta Y$ satisfy these bounds in the limit. \qed
Observation: Consider a Bernoulli $p$-random $X$ (obtained by flipping a coin with probability $p$ of getting a 1). We have $\dim(X) = H(p)$ and $\rho(X) = p$.

Obviously, we will need at least density $1/2 - p$ of changes to bring the density up to $1/2$, a necessary pre-requisite for bringing the effective dimension to 1.

Proposition: For each $s$, there is $X$ with $\dim(X) = s$ such that for all $Y$ with $\dim(Y) = 1$, we have $\overline{\rho}(X \Delta Y) \geq 1/2 - H^{-1}(s)$.

($X$ is any Bernoulli $H^{-1}(s)$-random.)

Theorem 3: For any $s < 1$ and any $X$ with $\dim(X) = s$, there is $Y$ with $\dim(Y) = 1$ and $d(X, Y) \leq 1/2 - H^{-1}(s)$. 

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A finite increasing theorem

**Fact:** For any $\sigma, s, t$ with $\dim(\sigma) = s < t \leq 1$, there is $\tau$ with $
abla(\sigma \Delta \tau) \leq H^{-1}(t) - H^{-1}(s)$ and $\dim(\tau) = t$.

(more basic Vereshchagin-Vitanyi theory)
The Main Lemma

Let $X = \sigma_1 \sigma_2 \ldots$ where $|\sigma_i| = i^2$.

Recall $s_i = \dim(\sigma_i | \sigma_1 \ldots \sigma_{i-1})$.

**Lemma:** Let $t_1, t_2, \ldots$, and $d_1, d_2 \ldots$ be any sequences satisfying for all $i$,

$$d_i = H^{-1}(t_i) - H^{-1}(s_i).$$

Then there is $Y = \tau_1 \tau_2 \ldots$ such that for all $i$,

$$t_i \leq \dim(\tau_i | \tau_1 \ldots \tau_{i-1}) \quad \text{and} \quad \rho(\sigma_i \Delta \tau_i) \leq d_i.$$

Proof: Uses Harper’s Theorem and compactness.
A convexity argument

Given $X = \sigma_1\sigma_2 \ldots$ with $\dim(X) = s$, we want to produce $Y = \tau_1\tau_2 \ldots$ with $\dim(Y) = 1$ and $d(X, Y) \leq 1/2 - H^{-1}(s)$.

Let $t_i = 1$ for all $i$. Let $d_i = 1/2 - H^{-1}(s_i)$. Let $Y$ be as guaranteed by the Main Lemma. Then

$$\dim(Y) = \liminf_{i} \sum_{k=1}^{i} \frac{|\tau_k|}{|\tau_1 \ldots \tau_i|} t_k = 1$$

$$d(X, Y) = \limsup_{i} \sum_{k=1}^{i} \frac{|\tau_k|}{|\tau_1 \ldots \tau_i|} (1/2 - H^{-1}(s_i))$$

$$\leq 1/2 - H^{-1} \left( \liminf_{i} \sum_{k=1}^{i} \frac{|\tau_k|}{|\tau_1 \ldots \tau_i|} s_i \right) = 1/2 - H^{-1}(s)$$

because $s_i \mapsto 1/2 - H^{-1}(s_i)$ is concave.
Increasing dimension $s$ to dimension $1$:

- Distance at least $1/2 - H^{-1}(s)$ may be needed to handle starting with a Bernoulli $H^{-1}(s)$-random.
- This distance suffices (construction).

Decreasing dimension $1$ to dimension $s$:

- Distance at least $H^{-1}(1 - s)$ is needed for information coding reasons.
- This distance suffices (construction).
Increasing dimension $s$ to dimension $t$:

- Distance at least $H^{-1}(t) - H^{-1}(s)$ may be needed to handle starting with a Bernoulli $H^{-1}(s)$-random.
- Construction breaks (convexity)

Decreasing dimension $t$ to dimension $s$:

- Distance at least $H^{-1}(t - s)$ is needed for information coding reasons.
- Construction breaks (even finite version)
Failure of convexity I (increasing from $s$ to $t$)

**Strategy:** Pump all information density up to $t$.

**Problem:** setting all $t_i = t$ in the Main Lemma, the map $s_i \mapsto d_i = H^{-1}(t_i) - H^{-1}(s_i)$ is not concave.

(on the board)
Failures of convexity II (increasing from $s$ to $t$)

**Strategy:** Constant distance. Let $d = H^{-1}(t) - H^{-1}(s)$, pump in as much information as possible within distance $d$.

**Problem:** setting all $d_i = d$ in the Main Lemma, the map $s_i \mapsto t_i = H(d_i + H^{-1}(s_i))$ is not convex (except at some small values of $s_i$).

(on the board)
Theorem 3+: For any $s < t \leq 1$ and any $X$ with $\dim(X) = s$, there is $Y$ with $\dim(Y) = t$ and $d(X, Y) \leq H^{-1}(t) - H^{-1}(s)$.

Proof uses the following strategy:

Given $s_i$, set $t_i$ so that $(s_i, t_i)$ lies on the line connecting $(s, t)$ and $(1, 1)$.

This produces a map $s_i \mapsto d_i$ which is concave!!

(on the board)
**Theorem 3+**: For any $s < t \leq 1$ and any $X$ with $\dim(X) = s$, there is $Y$ with $\dim(Y) = t$ and $d(X, Y) \leq H^{-1}(t) - H^{-1}(s)$.

Proof uses the following strategy:

Given $s_i$, set $t_i$ so that $(s_i, t_i)$ lies on the line connecting $(s, t)$ and $(1, 1)$.

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(on the board)

(seven derivatives later, including a partial derivative with respect to one of the parameters, we prove this map is concave.)
Problem: This map only works for pairs \((s, t)\) such that the map \(s_i \rightarrow d_i\) is decreasing at \(s\).

After some undergraduate calculus, these are exactly the pairs \((s, t)\) satisfying

\[
(1 - t)g'(t) \leq (1 - s)g'(s)
\]

where \(g = H^{-1}\).

(on board)

We see that the line toeing strategy fails for some small values of \(s\).
Constant distance strategy, reprise

We have already seen a strategy that only succeeds on some small values of $s$ – the constant distance strategy.

(only four derivatives needed to show that $s_i \mapsto t_i$ has the required convexity properties for small $s$!)
We have already seen a strategy that only succeeds on some small values of $s$ – the constant distance strategy.

(only four derivatives needed to show that $s_i \mapsto t_i$ has the required convexity properties for small $s$!)

After some undergraduate calculus, the pairs $(s, t)$ for which the constant distance strategy works are exactly those satisfying

$$(1 - t)g'(t) \geq (1 - s)g'(s)$$

where $g = H^{-1}$. 
Yes, I really meant that

Line toeing strategy works at \((s, t)\) if and only if

\[(1 - t)g'(t) \leq (1 - s)g'(s)\]

Constant distance strategy works at \((s, t)\) if and only if

\[(1 - t)g'(t) \geq (1 - s)g'(s)\]

where \(g = H^{-1}\).

For every \(s < t \leq 1\), there is a working strategy (there is a way to set the \(t_i, d_i\) in the Main Lemma so that by convexity, the resulting \(Y\) has the right effective dimension and the right distance from a given \(X\)).

This proves Theorem 3+.

This is too precise to be a coincidence!?
Summary of the talk

Increasing dimension \( s \) to dimension \( t \):
- Distance at least \( H^{-1}(t) - H^{-1}(s) \) may be needed to handle starting with a Bernoulli \( H^{-1}(s) \)-random.
- This distance suffices (construction)

Decreasing dimension \( t \) to dimension \( s \):
- Distance at least \( H^{-1}(t - s) \) is needed for information coding reasons.
- Construction breaks (even finite version)
- In fact, this distance is demonstrably too short.
Questions

Given $s < t < 1$, what is the minimum distance $d$ such that for every $X$ with $\dim(X) = t$, there is a $Y$ with $\dim(Y) = s$ and $d(X, Y) \leq d$?

Why do the line-toeing and constant-distance strategies dovetail so perfectly?