Computability and model-theoretic aspects of families of sets and its generalizations

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in celebration of the research work of Professor Rod Downey
Wehner’s family

**Theorem (Wehner, 1999).** The family

\[ \mathcal{W} = \{ \{n\} \oplus F : F \text{ is finite & } F \neq W_n \} \]

is (uniformly) c.e. in a degree \( x \) if and only if \( x > 0 \).
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is (uniformly) c.e. in a degree \( x \) if and only if \( x > 0 \).

**Corollary.** There is a countable algebraic structure \( \mathcal{A} \) s.t. \( \mathcal{A} \) has an \( x \)-computable structure if and only if \( x > 0 \).
Jump inversions

**Theorem** (Goncharov, Harizanov, Knight, McCoy, Miller, Solomon, 2005). There is a (strong) jump inversion in the class of structures, i.e. functor $F$ s.t.

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$A$ has an $x'$-comp. copy $\iff F(A)$ has an $x$-comp. copy.

Corollary. For every $n \in \omega$ there is a countable algebraic structure $A$ s.t. $A$ has an $x$-computable structure if and only if $x^{(n)} > 0^{(n)}$. 

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\mathcal{A} \text{ has an } x'-\text{comp. copy } \iff F(\mathcal{A}) \text{ has an } x\text{-comp. copy.}
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**Corollary.** For every \( n \in \omega \) there is a countable algebraic structure \( \mathcal{A} \) s.t. \( \mathcal{A} \) has an \( x \)-computable structure if and only if \( x^{(n)} > 0^{(n)} \).

**Proof.** Let \( \mathcal{B} \) has an \( x \)-computable copy iff \( x > 0^{(n)} \).
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**Corollary.** For every $n \in \omega$ there is a countable algebraic structure $\mathcal{A}$ s.t. $\mathcal{A}$ has an $x$-computable structure if and only if $x^{(n)} > 0^{(n)}$.

**Proof.** Let $\mathcal{B}$ has an $x$-computable copy iff $x > 0^{(n)}$. Then $\mathcal{A} = F^n(\mathcal{B})$. 
Iterated jump inversions

**Theorem** (Goncharov, Harizanov, Knight, McCoy, Miller, Solomon, 2005). There are iterated jump inversions for successive computable ordinals \( \alpha \), i.e. functors \( F^{(\alpha)} \) s.t.

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A \text{ has an } x^{(\alpha)}\text{-comp. copy } \iff F^{(\alpha)}(A) \text{ has an } x\text{-comp. copy.}
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$\mathcal{A}$ has an $x^{(\alpha)}$-comp. copy $\iff F^{(\alpha)}(\mathcal{A})$ has an $x$-comp. copy.

**Corollary.** For every successive $\alpha \in \omega_1^{CK}$ there is a countable algebraic structure $\mathcal{A}$ s.t. $\mathcal{A}$ has an $x$-computable structure if and only if $x^{(\alpha)} > 0^{(\alpha)}$. 
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**Proof.** Let $B$ has an $x$-computable copy iff $x > 0^{(\alpha)}$. Then $A = F^{(\alpha)}(B)$. 

Kalimullin I.Sh. Computability and model-theoretic aspects of families
JUMP INVERSIONS AND FAMILIES

INFINITELY ITERATED JUMP INVERSIONS

LEAST JUMP INVERSIONS

Jump inversion for families

**Question.** Are there jump inversion functors for families? Are there families which are uniformly $x$-c.e. if and only if $x^{(\alpha)} > 0^{(\alpha)}$?
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**Answer:** Only for $\alpha = 0$ and $\alpha = 1$ (K., Faizrahmanov, 2015).
The family enumerable in non-low₁ degrees

**Theorem.** (Andrews, Cai, K, Lempp, Miller, Montalban, 2016). Let $\emptyset' \equiv_T \delta \in \omega^\omega$ such that the set

$$C = \{ \sigma \in \omega^{<\omega} \mid \sigma \not\subseteq \delta \}$$

is c.e. Then the family

$$\mathcal{V} = \{ \{n\} \oplus (C \cup F) \mid F \text{ is finite and } F \neq W_n^\delta \}$$

is $x$-c.e. iff $x \not\leq 0'$. 
The family enumerable in non-low₁ degrees

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is c.e. Then the family

\[ \mathcal{V} = \{ \{ n \} \oplus (C \cup F) | F \text{ is cofinite and } \overline{F} \neq W_n^\delta \} \]

is $x$-c.e. iff $x' > 0'$. 
The case \( \alpha = 2 \)

**Theorem.** (Faizrahmanov, K.) There is no family \( \mathcal{F} \) which is \( x \)-c.e. iff \( x'' > 0'' \).
The case $\alpha = 2$

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**Theorem.** (Faizrahmanov, K.) There is no family $\mathcal{F}$ which is $x$-c.e. iff $x'' > 0''$.

**Proof.** Let $X$ be a low$_3$ c.e. set which is not low$_2$. Then the index set $\{e : \Phi^X_e \text{ is non-low}_2\}$ is $(\Pi^X_5 = \Pi_5)$-complete.
The case \( \alpha = 2 \)

**Theorem.** (Faizrahmanov, K.) There is no family \( \mathcal{F} \) which is \( x \)-c.e. iff \( x'' > 0'' \).

**Proof.** Let \( X \) be a low\(_3\) c.e. set which is not low\(_2\). Then the index set \( \{ e : \Phi^X_e \text{ is non-low}_2 \} \) is \( (\Pi^X_5 = \Pi_5) \)-complete. But if \( \mathcal{F} \) is \( X \)-c.e. then the index set \( \{ e : \mathcal{F} \text{ is } \Phi^X_e \text{-c.e.} \} \) is \( \Sigma^X_5 = \Sigma_5 \).
Definition. A 0-family is any subset of $\omega$. 
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An $n$-family, $0 < n < \omega$, is a countable set of $m$-families, $m < n$. 
Generalizations of families for jump inversions

**Definition.** A **0-family** is any subset of $\omega$.

An **$n$-family**, $0 < n < \omega$, is a countable set of $m$-families, $m < n$.

An $n$-family $\mathcal{U}$ is **$x$-c.e.** if the $m$-families $\mathcal{V} \in \mathcal{U}$, $m < n$, are uniformly $x$-c.e.
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An n-family \( \mathcal{U} \) is \( x \)-c.e. if the m-families \( \mathcal{V} \in \mathcal{U}, m < n \), are uniformly \( x \)-c.e.

Observation. Every n-family \( \mathcal{U} \) can be coded into a structure \( \mathcal{G}_\mathcal{U} \) such that \( \mathcal{U} \) is \( x \)-c.e. iff \( \mathcal{G}_\mathcal{U} \) has a \( x \)-computable copy.
Jump inversion for \( n \)-families

An \( n \)-family \( \mathcal{U} \) is \( x' \)-c.e. iff the \((n + 1)\)-family

\[
\mathcal{E}(\mathcal{U}) = \begin{cases} 
\{\omega\} \cup \{\{x\} : x \in A\}, & \text{if } n = 0 \text{ and } \mathcal{U} = A \subseteq \omega, \\
\{\mathcal{E}(\mathcal{V}) : \mathcal{V} \in \mathcal{U}\}, & \text{if } n > 0,
\end{cases}
\]

is \( x \)-c.e.
Double jump inversion for \( n \)-families

An \( n \)-family \( \mathcal{U} \) is \( x'' \)-c.e. iff the \((n + 1)\)-family

\[
\mathcal{D}(\mathcal{U}) = \begin{cases} 
\{\text{all finite sets}\} \cup \{\overline{x} : x \in A\}, & \text{if } n = 0 \text{ and } \mathcal{U} = A \subseteq \omega, \\
\{\mathcal{D}(\mathcal{V}) : \mathcal{V} \in \mathcal{U}\}, & \text{if } n > 0,
\end{cases}
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Double jump inversion for $n$-families

An $n$-family $\mathcal{U}$ is $x''$-c.e. iff the $(n + 1)$-family

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\{\mathcal{D}(\mathcal{V}) : \mathcal{V} \in \mathcal{U}\}, & \text{if } n > 0,
\end{cases}$$

is $x$-c.e.

**Theorem** (Faizrahmanov, K., 2015) For every $n \in \omega$ there are $(n + 1)$-families $\mathcal{U}_n$ and $\mathcal{V}_n$ such that

$$\mathcal{U}_n \text{ is } x\text{-c.e. } \iff x^{(2n)} > O^{(2n)}$$

and

$$\mathcal{V}_n \text{ is } x\text{-c.e. } \iff x^{(2n+1)} > O^{(2n+1)}.$$
Definition. A 0-family is any subset of $\omega$.

An $\alpha$-family, $0 < \alpha < \omega_1^{CK}$, is a countable set of $\beta$-families, $\beta < \alpha$.

An $\alpha$-family $\mathcal{U}$ is $x$-c.e. if the $\beta$-families $\mathcal{V} \in \mathcal{U}$, $\beta < \alpha$, are uniformly $x$-c.e.
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Generalized families for infinitely iterated jump inversions

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**Observation.** Every $\alpha$-family $\mathcal{U}$ can be coded into a structure $\mathcal{G}_\mathcal{U}$ such that $\mathcal{U}$ is $\mathbf{x}$-c.e. iff $\mathcal{G}_\mathcal{U}$ has a $\mathbf{x}$-computable copy.
The \((\omega + 1)\)-jump inversion

Let \(\mathcal{E}^\omega(A)\) be the \((\omega + 1)\)-family containing all \(\omega\)-families in the form

\[
\{\mathcal{E}^n(L(n)) : n \in \omega\},
\]

where \(L : \omega \to 2^\omega\) is any function such that \(L(n)\) is finite for every \(n\) and beginning some \(n\) we have

\[
L(n) = L(n + 1) \subseteq A.
\]

**Theorem.** (Faizrahmanov, K., 2016) A set \(A\) is \(x^{(\omega + 1)}\)-c.e. iff the \((\omega + 1)\)-family \(\mathcal{E}^{\omega + 1}(A)\) is \(x\)-c.e.
The non-low\(\omega\) and non-low\(\omega+1\) degrees

**Corollary** (Faizrahmanov, K., 2016). There is an \((\omega + 2)\)-family \(\mathcal{U}\) such that

\[ \mathcal{U} \text{ is } x\text{-c.e.} \iff x^{(\omega+1)} > 0^{(\omega+1)}. \]
The non-low\(_\omega\) and non-low\(_{\omega+1}\) degrees

**Corollary** (Faizrahmanov, K., 2016). There is an \((\omega + 2)\)-family \(U\) such that

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Corollary. There is an algebraic structure $\mathcal{A}$ such that

$\mathcal{A} \text{ has an x-comp. copy } \iff x^{(\omega)} > 0^{(\omega)}.$
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Theorem. (Kach, K., Montalban, Soskov, 2012, unpublished). There is no algebraic structure $\mathcal{A}$ such that

$$\mathcal{A} \text{ has an } x\text{-comp. copy } \iff x^{(\omega)} \geq a$$

if $a > 0^{(\omega)}$. 
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if $a > 0^{(\omega)}$.

**Theorem.** (Soskov, 2013). There is a structure $\mathcal{B}$ such that for no algebraic structure $\mathcal{A}$ such that

$$(\exists x)[y = x^{(\omega)} \& \mathcal{A} \text{ has an } x\text{-comp. copy}]$$

$$\iff y \geq 0^{(\omega)} \& \mathcal{B} \text{ has an } y\text{-comp. copy.}$$
$\alpha$-jump inversion

**Theorem.** (Faizrahmanov, K., 2016) For a set $A$ and successive $\alpha < \omega_1^{CK}$ one can define an $\alpha$-family $E^\alpha(A)$ such that $A$ is $x^{(\alpha)}$-c.e. iff $E^\alpha(A)$ is $x$-c.e.
\(\alpha\)-jump inversion

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**Corollary** (Faizrahmanov, K., 2016). For a successive \(\alpha < \omega_1^{CK}\) there is an \((\alpha + 1)\)-family \(U\) such that

\[U \text{ is } x\text{-c.e. } \iff x^{(\alpha)} > 0^{(\alpha)}.\]
\( \alpha \)-jump inversion

**Theorem.** (Faizrahmanov, K., 2016) For a set \( A \) and successive \( \alpha < \omega_1^{CK} \) one can define an \( \alpha \)-family \( E^\alpha(A) \) such that \( A \) is \( x^{(\alpha)} \)-c.e. iff \( E^\alpha(A) \) is \( x \)-c.e.

**Corollary** (Faizrahmanov, K., 2016). For a successive \( \alpha < \omega_1^{CK} \) there is an \( (\alpha + 1) \)-family \( U \) such that
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U \text{ is } x \text{-c.e. } \iff x^{(\alpha)} > 0^{(\alpha)}.
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**Theorem** (Faizrahmanov, K., 2016). For a limit \( \alpha < \omega_1^{CK} \) there is an \( (\alpha + 1) \)-family \( U \) such that
\[
U \text{ is } x \text{-c.e. } \iff x^{(\alpha)} > 0^{(\alpha)}.
\]
**α-jump inversion**

**Theorem.** (Faizrahmanov, K., 2016) For a set $A$ and successive $\alpha < \omega_1^{CK}$ one can define an $\alpha$-family $E^\alpha(A)$ such that $A$ is $x^{(\alpha)}$-c.e. iff $E^\alpha(A)$ is $x$-c.e.

**Corollary** (Faizrahmanov, K., 2016). For a successive $\alpha < \omega_1^{CK}$ there is an $(\alpha + 1)$-family $U$ such that

\[ U \text{ is } x\text{-c.e.} \iff x^{(\alpha)} > 0^{(\alpha)}. \]

**Theorem** (Faizrahmanov, K., 2016). For a limit $\alpha < \omega_1^{CK}$ there is an $(\alpha + 1)$-family $U$ such that

\[ U \text{ is } x\text{-c.e.} \iff x^{(\alpha)} > 0^{(\alpha)}. \]

**Corollary.** For every $\alpha < \omega_1^{CK}$ there is an algebraic structure $A$ such that

\[ A \text{ has an } x\text{-comp. copy} \iff x^{(\alpha)} > 0^{(\alpha)}. \]
Least jump inversions

A \leq _\Sigma B \text{ means that } A \text{ is } \Sigma_{c1} \text{-interpretable in } B^{<\omega}.

\( A(\alpha) = (A, \text{all } \Sigma_{c\alpha} \text{-predicates}) \).

A countable structure \( B \) is a least \( \alpha \)-jump inversion for a countable structure \( A \) if \( A \leq _\Sigma X(\alpha) \iff B \leq _\Sigma X \) for every countable structure \( X \).

(Example). The family of all infinite c.e. sets is a least jump inversion for \( 0'' \), but the family of all total computable functions is not.
Least jump inversions

- $A \leq_{\Sigma} B$ means that $A$ is $\Sigma^c_1$-interpretable in $B^{<\omega}$.
Least jump inversions

- $\mathcal{A} \leq_{\Sigma} \mathcal{B}$ means that $\mathcal{A}$ is $\Sigma^c_{1}$-interpretable in $\mathcal{B}^{<\omega}$.
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A \leq_{\Sigma} \mathcal{X}^{(\alpha)} \iff B \leq_{\Sigma} \mathcal{X}
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Least jump inversions

- $\mathcal{A} \leq_{\Sigma} \mathcal{B}$ means that $\mathcal{A}$ is $\Sigma^c_1$-interpretable in $\mathcal{B}^{<\omega}$.
- $\mathcal{A}^{(\alpha)} = (\mathcal{A}, \text{all } \Sigma^c_\alpha\text{-predicates})$.
- A countable structure $\mathcal{B}$ is a least $\alpha$-jump inversion for a countable structure $\mathcal{A}$ if
  \[ \mathcal{A} \leq_{\Sigma} \mathcal{X}^{(\alpha)} \iff \mathcal{B} \leq_{\Sigma} \mathcal{X} \]
  for every countable structure $\mathcal{X}$.
- (Example). The family of all infinite c.e. sets is a least jump inversion for $0''$, but the family of all total computable functions is not.
Existence of least jump inversions

**Theorem** (Faizrahmanov, K., Montalban, Puzarenko). For every successive $\alpha < \omega_1^{CK}$ and every countable structure $\mathcal{A}$ there is a least jump inversion $\mathcal{A}^{(-\alpha)}$. 
Existence of least jump inversions

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By the definition $A \leq \Sigma B$ implies $A^{(-\alpha)} \leq \Sigma B^{(-\alpha)}$. 
Existence of least jump inversions

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By the definition $A \leq \Sigma B$ implies $A^{(-\alpha)} \leq \Sigma B^{(-\alpha)}$. By the construction $(A \oplus B)^{(-\alpha)} \leq \Sigma A^{(-\alpha)} \oplus B^{(-\alpha)}$. 
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By the definition $A \leq_{\Sigma} B$ implies $A^{(-\alpha)} \leq_{\Sigma} B^{(-\alpha)}$. By the construction $(A \oplus B)^{(-\alpha)} \leq_{\Sigma} A^{(-\alpha)} \oplus B^{(-\alpha)}$. Thus,

**Corollary.** $(A \oplus B)^{(-\alpha)} \equiv_{\Sigma} A^{(-\alpha)} \oplus B^{(-\alpha)}$. 
Least jump inversion for generalized families

**Theorem** (Faizrahmanov, K., Montalban, Puzarenko). For a set $A$ the family $E(A) = \{\omega\} \cup \{\{x\} : x \in A\}$ is the least jump inversion for $A$. Thus, the least jump inversion for an $\beta$-family is an $(1 + \beta)$-family.
Theorem (Faizrahmanov, K., Montalban, Puzarenko). For a set $A$ the family $\mathcal{E}(A) = \{\omega\} \cup \{\{x\} : x \in A\}$ is the least jump inversion for $A$. Thus, the least jump inversion for an $\beta$-family is an $(1 + \beta)$-family.

Remark. For a set $A$ the family $\mathcal{D}(A) = \{\text{all finite sets}\} \cup \{\{x\} : x \in A\}$ is not the least double jump inversion for $A$. Moreover, the $2$-family $\mathcal{E}(\mathcal{E}(A))$ is not $\Sigma$-equivalent to a $1$-family.
Theorem (Faizrahmanov, K., Montalban, Puzarenko). For a set $A$ the family $E(A) = \{\omega\} \cup \{\{x\} : x \in A\}$ is the least jump inversion for $A$. Thus, the least jump inversion for an $\beta$-family is an $(1 + \beta)$-family.

Remark. For a set $A$ the family $D(A) = \{\text{all finite sets}\} \cup \{\{x\} : x \in A\}$ is not the least double jump inversion for $A$. Moreover, the $2$-family $E(E(A))$ is not $\Sigma$-equivalent to a $1$-family.

Theorem. For a set $A$ and successive $\alpha < \omega_1^{CK}$ the $\alpha$-family $E^\alpha(A)$ is the least $\alpha$-jump inversion for $A$. Thus, the least $\alpha$-jump inversion for an $\beta$-family is an $(\alpha + \beta)$-family.