Effective fractal dimension theory: exploring the extreme cases

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Effective dimension

- Lutz defines effective dimension as a generalization of the classical notion of fractal dimension
- This gives very **robust concepts**, they can be defined using
  - measure theory
  - gambling
  - information theory
- **Resource-bounded versions** are natural and useful quantitative tools
- Effectivization of Hausdorff dimension gives a **partial randomness concept**

This mini course

0. **Introduction of effective dimension**
1. Resource-bounded Hausdorff dimension for Complexity Classes
2. Compression and dimension for low resource bounds. Very effective construction of a normal sequence
3. Looking back at fractal geometry, other metric spaces

Warning: references mostly at the end of each lecture

Dimension in fractal geometry

- Hausdorff dimension is defined in every metric space X
- Every set $A \subseteq X$ is associated a dimension $s \in [0, \infty)$
- It is a powerful quantitative tool:
  - "Probabilistic" method
    $\dim(A) > 0$ implies $A \neq \emptyset$; $\dim(A^c) < \dim(X)$ implies $A \neq \emptyset$
  - Abundance proofs ($\dim(A) > 0$ is far stronger than $A \neq \emptyset$)
  - New hypothesis (Assume $\dim(A) > 0$ and prove results that did not seem to follow from weaker hypothesis)
- In Euclidean space, this concept coincides with our intuition that smooth curves have dimension 1 and smooth surfaces have dimension 2, but from its introduction in 1918 Hausdorff noted that many sets have noninteger dimension, what he called "fractional dimension"
- In the 1980s Tricot and Sullivan independently developed a dual of Hausdorff dimension called packing dimension

Algorithmic randomness

- Can we generate randomness?
- Can we quantify randomness?
- What can we compute using randomness?

Definitions of algorithmic randomness: three approaches

- the measure theory approach: Abundance/tipicality. Random sequences should not have effectively rare properties (von Mises, 1919, finally Martin-Löf 1966)
- the gambler's approach: Unpredictability. A betting strategy can exploit rare patterns. Random sequences should be unpredictable. (Solomonoff, 1961, Scnhorr, 1975, Levin 1970)
- the information theory approach: Uncompressibility. Random sequences should not be compressible (i.e., easily describable) (Kolmogorov, Levin, Chaitin 1960-1970's)

Partial randomness: Effective dimension

- Effectivization of Hausdorff dimension gives a partial randomness concept
- Martin-Löf random sequences have effective dimension 1
- Every sequence (and set of sequences) has an effective dimension between 0 and 1 (end of nonmeasurability)
- Robust concept: can be defined in terms of gambling and Kolmogorov complexity/compressibility ratio
- Effective fractal dimension is a measure of information content providing the typicalness and predictability intuitions

Computational Complexity: resource-bounds on randomness

- Lutz resource-bounded measure and randomness: it can be adapted to each Complexity Class to have a meaningful/useful concept of effective measure/randomness
- Very low resource-bounds still give meaningful concepts
- Normality corresponds to constant memory randomness (or finite-state randomness)
- In some interesting cases it is definable using both prediction and compression (pspace, FS)
- It inherits non measurability issues from Martin-Löf approach

Computational Complexity: resource-bounded dimension

- Lutz resource-bounded dimension: it can be adapted to each Complexity Class to have a meaningful/useful concept of effective dimension
- Very low resource-bounds still give meaningful concepts
- Normality corresponds to constant memory dimension 1 (or maximal finite-state dimension)
- In most interesting cases it is definable using both prediction and compression
- Every set in assigned an effective dimension

Let us move to definitions ...

Our notation for Cantor space

- For Σ a finite alphabet, Σ^* is the set of finite sequences over Σ ($\{0,1\}^*$)
- $\{0,1\}^\infty$ is the set of infinite binary sequences
- For $x \in \{0,1\}^\infty$, $x \upharpoonright n$ the the length n finite prefix of x

- In Computational Complexity we will identify a problem/language $A \subseteq \{0,1\}^*$ with its characteristic (infinite) sequence $\chi_A \in \{0,1\}^\infty$
- Otherwise we may be interested in the real number in $[0,1]$ represented by each $x \in \{0,1\}^\infty$ (the number with binary representation $0.x$) the choice of alphabet can be relevant

Lutz gambling characterization of dimension in Cantor space

- For $s \in [0, \infty)$, an s-supergale is a function $d : \{0,1\}^* \to [0, \infty)$ such that $w \in \{0,1\}^*$

$$d(w) \geq \frac{d(w0) + d(w1)}{2^s}$$

- The success set of an s-supergale d is

$$S^\infty[d] = \left\{ x \in \{0,1\}^\infty \,\middle|\, \limsup_n d(x \upharpoonright n) = \infty \right\}$$

Theorem

For every $A \subseteq \{0,1\}^\infty$,

$$\dim_{\mathrm{H}}(A) = \inf \{ s \,|\, \text{there is an } s\text{-supergale } d \text{ such that } A \subseteq S^\infty[d] \}$$

Variants

- Use $\liminf$ in the success definition:
  $S^{\infty}_{\mathrm{str}}[d] = \{x \in \{0,1\}^{\infty} \mid \liminf_n d(x \upharpoonright n) = \infty\}$ to characterize packing dimension
- Use martingale growth rates in the place of gales
- gales or supergales

Constructive dimension in Cantor space

The constructive dimension of A is

$$\mathrm{cdim}(A) = \inf \left\{ s \;\middle|\; \begin{array}{l} \text{there is a constructive } s\text{-supergale } d \\ \text{such that } A \subseteq S^\infty[d] \end{array} \right\}$$

Constructive means lower semi-computable, that is d is constructive if there is an exactly computable function $\hat{d} : \Sigma^* \times \mathbb{N} \to \mathbb{Q}$ with the following two properties.

- For all $w \in \Sigma^*$ and $t \in \mathbb{N}$, $\hat{d}(w,t) \leq \hat{d}(w,t+1) < d(w)$.
- For all $w \in \Sigma^*$, $\lim_{t\to\infty} \hat{d}(w,t) = d$.

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Resource-bounded dimensions

Let Δ be a class of functions (e.g., polynomial time computable, polynomial space computable)

The Δ-dimension of A is

$$\dim_\Delta(A) = \inf \left\{ s \;\middle|\; \begin{array}{l} \text{there is an } s\text{-supergale } d \in \Delta \\ \text{such that } A \subseteq S^\infty[d] \end{array} \right\}$$

- Choosing different Δ we restrict gales to different classes of computable strategies
- With gales computable by a finite automata we get $\dim_{\mathrm{FS}}$
- $\dim_{\mathrm{p}}$ corresponds to computable in polynomial time
- $\dim_{\mathrm{pspace}}$ means polynomial space computable gales

Each of this effective dimensions is “the right one” for a set of sequences (complexity class)

Complexity classes

Each r-b dimension is the right one for a complexity class

- $\mathrm{E} = \mathrm{DTIME}(2^{O(n)})$, we have $\dim_{\mathrm{p}}(\mathrm{E}) = 1$
- $\mathrm{EXP} = \mathrm{DTIME}(2^{n^{O(1)}})$, p_2 is 2^{polylog} time computable, we have $\dim_{\mathrm{p}_2}(\mathrm{EXP}) = 1$
- $\mathrm{ESPACE} = \mathrm{DSPACE}(2^{O(n)})$, we have $\dim_{\mathrm{pspace}}(\mathrm{ESPACE}) = 1$
- $\mathrm{EXPSPACE} = \mathrm{DSPACE}(2^{n^{O(1)}})$, $\mathrm{p}_2\mathrm{space}$ is 2^{polylog} space computable, we have $\dim_{\mathrm{p}_2\mathrm{space}}(\mathrm{EXPSPACE}) = 1$
- $\dim_{\mathrm{FS}}(\mathbb{Q}) = 1$

Sometimes we denote $\dim_{\mathrm{p}}(X \cap E)$ as "dimension in E of X", etc.

Some properties or resource-bounded dimension

- $\dim_\Delta(X)$ is defined for every set X
- $X \subseteq Y$ implies $\dim_\Delta(X) \leq \dim_\Delta(Y)$
- $\dim_\Delta(\cup_i X_i) = \sup_i \dim_\Delta(X_i)$ for "suitable" effective unions

where $\dim_\Delta$ is any of the effective dimensions

Uses of effective dimension in complexity

- Abundance proofs
- Probabilistic method
- New hypothesis, new concepts

A taste of r.b. dimension: Abundance proofs

- The class of sets that (polynomial-time) reduce to a nondense set has p-dimension 0 in Exponential time (E)
- E has p-dimension 1
- **Most sets in Exponential time do not reduce to a nondense set**

Abundance result in detail

How dense are hard sets for exponential time?

- The most common notions of polynomial time reductions are many-one $\leq_{\mathrm{m}}^{\mathrm{p}}$ and Turing $\leq_{\mathrm{T}}^{\mathrm{p}}$
- In between $\leq_{\mathrm{m}}^{\mathrm{p}}$ and $\leq_{\mathrm{T}}^{\mathrm{p}}$ is a wide variety of polynomial-time reductions of different strengths
- Reductions are often used to prove hardness for a complexity class, we will look at E and EXP

$$\mathrm{DENSE} = \left\{ L \,\middle|\, \exists \epsilon \,\dot{\forall} n |L^{\leq n}| > 2^{n^{\epsilon}} \right\}$$

- All known hard problems for E and EXP are dense
- **Is every hard se dense?**

Density of hard sets

Known:

- (Watanabe 1987) Every hard set for E under the $\leq^{\mathrm{p}}_{\log-\mathrm{tt}}$ reductions is dense
- (Lutz Mayordomo 1994) Every hard set for E under the $\leq^{\mathrm{p}}_{n^\alpha-\mathrm{tt}}$ $(\alpha < 1/2)$ reductions is dense
- (Fu 1995, Lutz Zhao 2000) Every hard set for E under the $\leq^{\mathrm{p}}_{n^\alpha-\mathrm{T}}$ $(\alpha < 1/2)$ reductions is dense. Every hard set for EXP under the $\leq^{\mathrm{p}}_{n^\alpha-\mathrm{T}}$ $(\alpha < 1)$ reductions is dense

Curious contrast E, EXP ...

Density of hard sets: abundance result

- (Hitchcock 2005, Harkins Hitchcock 2011) improved all previous results by showing the following result

Theorem

The p-dimension of sets that reduce to nondense sets (under $\leq^{\mathrm{p}}_{n^\alpha-\mathrm{T}}$ $(\alpha < 1)$ reduction) is 0

- Their proof is quite involved, including:
  - the online mistake-bound model of learning
  - reduction to learnable concepts
  - the set of reducible to learnable concepts has p-dimension 0
  - sets that reduce to nondense are reducible to learnable classes (monotone disjunctions with few literals)
- Abundance result $(\dim_{\mathrm{p}}(\mathrm{E}) = 1)$ Most sets in E do not reduce to nondense sets
- Existence result (probabilistic method) There is a set in E that does not $\leq^{\mathrm{p}}_{n^\alpha-\mathrm{T}}$-reduce $(\alpha < 1)$ to nondense sets
- Consequence: All $\leq^{\mathrm{p}}_{n^\alpha-\mathrm{T}}$-hard sets for E are dense

A taste of effective dimension: Probabilistic method

- $\dim_p(absly - normal) = 1$ (The set of absolutely normal numbers have polynomial-time dimension 1)
- A real number α is normal in base b (Borel 1909) if the base b representation of α for every finite sequence w of base b digits the asymptotic, empirical frequency of w in the base-b expansion of α is $b^{|w|}$
- Absolutely-normal number means normal in every base
- The result implies an efficient way of constructing an absolutely normal real number (**constructive probabilistic method**)

I will get back to this in my next lecture

A taste of effective dimension: new hypothesis

- It is not known whether all NP-hard sets are dense
- If $\dim_{\mathrm{p}}(\mathrm{NP}) > 0$ then all $\leq^{\mathrm{p}}_{n^{\alpha}-\mathrm{T}}$-hard sets for NP are dense

A taste of effective dimension: new hypothesis

- MAX3SAT is the problem of computing the number of satisfied clauses in a 3SAT formula
- If $\dim_{\mathrm{p}}(NP) > 0$ then MAX3SAT is hard to approximate (effective approximation algorithms have performance ration less than $7/8$ on a dense set of instances)

A taste of effective dimension: new hypothesis

- BPP is the class of problems solvable in bounded error probabilistic polynomial time
- **Zero-One law**: $\dim_{\mathrm{p}_2}(\mathrm{BPP}) = 0$ or $\mathrm{BPP} = \mathrm{EXP}$

Resource-bounded dimension: changing the scale

- Let $g : \mathbb{N} \times [0, \infty) \to [0, \infty)$ be a scale function (a family of gauge functions).
- Usual Hausdorff dimension corresponds to the scale $g(m, s) = sm$
- For $s \in [0, \infty)$, an g-s-supergale is a function $d : \{0, 1\}^* \to [0, \infty)$ such that $w \in \{0, 1\}^*$

$$d(w) \geq \frac{d(w0) + d(w1)}{2^{g(|w|+1,s)-g(|w|,s)}}$$

- The success set of an g-s-supergale d is

$$S^\infty[d] = \left\{ x \in \{0, 1\}^\infty \middle| \limsup_n d(x \upharpoonright n) = \infty \right\}$$

Theorem

For every $A \subseteq \{0, 1\}^\infty$,

$$\dim_{\mathrm{H}}^g(A) = \inf \{ s \,|\, \textit{there is a } g\text{-}s\textit{-supergale } d \textit{ such that } A \subseteq S^\infty[d] \}$$

Resource-bounded dimension: changing the scale

- Related to the classical concept of exact or general dimension
- We consider different scales g for which $\dim_{\mathrm{p}}^{g}(\mathrm{E}) = 1$, $\dim_{\mathrm{pspace}}^{g}(\mathrm{ESPACE}) = 1$
- For certain scales g, g' it holds that that

$$\dim_{\mathrm{pspace}}^{g}(\mathrm{SIZE}(2^{\alpha n})) = \alpha$$

$$\dim_{\mathrm{pspace}}^{g'}(\mathrm{SIZE}(2^{n^{\alpha}})) = \alpha$$

A taste of effective dimension: small span theorems

- Small span theorems: given a reduction, either the upper or the lower span is small
- For a language A and a reduction r, the upper spam is

$$\mathrm{P}_r^{-1}(A) = \{B \,|A \leq_r^{\mathrm{p}} B\}$$

- For a language A and a reduction r, the lower spam is

$$\mathrm{P}_r(A) = \{B \,|B \leq_r^{\mathrm{p}} A\}$$

Theorem

For every A in E, either

$$\dim_{\mathrm{p}}^{g}(\mathrm{P}_{\mathrm{m}}(A)) = 0$$

or

$$\dim_{\mathrm{p}}^{g}(\mathrm{P}_{\mathrm{m}}^{-1}(A)) = 0$$

Open questions on resource-bounded dimension

- What is the p-dimension of NP?
- Is it possible that $0 < \dim_{\mathrm{p}}(\mathrm{NP}) < 1$?

Open questions: Partially complete problems

- A problem X is complete for a class C if every $Z \in C$ can be reduced to X
- A problem X is *partially complete* for a class C if the set of $Z \in C$ that can be reduced to X has nonzero dimension in C.
- OPEN:
  - Examples of natural partially complete problems
  - Is Graph Isomorphism partially complete for EXP?
  - Are partially complete the same for E and EXP?

Open questions: Finding sources for BPP

- In certain ways positive dimension can substitute Martin-Löf randomness
- It was known that for each Martin-Lf random x, $\mathrm{BPP} \subseteq \mathrm{P}^x$ (in fact for much lower resource-bounded randomness)
- Can I have $\mathrm{P}^A = \mathrm{BPP}$ when $\dim_{\mathrm{p}}(A) > 0$?

References for resource-bounded dimension

Main references

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Next lecture

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