Effective fractal dimension theory: exploring the extreme cases

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Effective dimension

- Lutz defines effective dimension as a generalization of the classical notion of fractal dimension.
- This gives very robust concepts, they can be defined using measure theory, gambling, and information theory.
- Resource-bounded versions are natural and useful quantitative tools.
- Effectivization of Hausdorff dimension gives a partial randomness concept.
This mini course

0. **Introduction of effective dimension**
1. Resource-bounded Hausdorff dimension for Complexity Classes
2. Compression and dimension for low resource bounds. Very effective construction of a normal sequence
3. Looking back at fractal geometry, other metric spaces

Warning: references mostly at the end of each lecture
**Dimension in fractal geometry**

- **Hausdorff dimension** is defined in every metric space $X$.
- Every set $A \subseteq X$ is associated a dimension $s \in [0, \infty)$.
- It is a powerful **quantitative tool**:
  - “Probabilistic” method
    \[ \dim(A) > 0 \text{ implies } A \neq \emptyset; \quad \dim(A^c) < \dim(X) \text{ implies } A \neq \emptyset \]
  - Abundance proofs ($\dim(A) > 0$ is far stronger than $A \neq \emptyset$)
  - New hypothesis (Assume $\dim(A) > 0$ and prove results that did not seem to follow from weaker hypothesis)

- In Euclidean space, this concept coincides with our intuition that smooth curves have dimension 1 and smooth surfaces have dimension 2, but from its introduction in 1918 Hausdorff noted that many sets have noninteger dimension, what he called “fractional dimension”.

- In the 1980s Tricot and Sullivan independently developed a dual of Hausdorff dimension called packing dimension.
Algorithmic randomness

- Can we generate randomness?
- Can we quantify randomness?
- What can we compute using randomness?
Definitions of algorithmic randomness: three approaches

- **the measure theory approach**: Abundance/tipicality. Random sequences should not have effectively rare properties (von Mises, 1919, finally Martin-Löf 1966)

- **the gambler’s approach**: Unpredictability. A betting strategy can exploit rare patterns. Random sequences should be unpredictable. (Solomonoff, 1961, Schnorr, 1975, Levin 1970)

- **the information theory approach**: Uncompressibility. Random sequences should not be compressible (i.e., easily describable) (Kolmogorov, Levin, Chaitin 1960-1970’s)
Partial randomness: Effective dimension

- Effectivization of Hausdorff dimension gives a partial randomness concept
- Martin-Löf random sequences have effective dimension 1
- Every sequence (and set of sequences) has an effective dimension between 0 and 1 (end of nonmeasurability)
- Robust concept: can be defined in terms of gambling and Kolmogorov complexity/compressibility ratio
- Effective fractal dimension is a measure of information content providing the typicalness and predictability intuitions
Computational Complexity: resource-bounds on randomness

- **Lutz resource-bounded measure and randomness**: it can be adapted to each Complexity Class to have a meaningful/useful concept of effective measure/randomness.
- Very low resource-bounds still give meaningful concepts.
- Normality corresponds to constant memory randomness (or finite-state randomness).
- In some interesting cases it is definable using both prediction and compression (pspace, FS).
- It inherits non measurability issues from Martin-Löf approach.
Computational Complexity: resource-bounded dimension

- **Lutz resource-bounded dimension**: it can be adapted to each Complexity Class to have a meaningful/useful concept of effective dimension
- Very low resource-bounds still give meaningful concepts
- Normality corresponds to constant memory dimension 1 (or maximal finite-state dimension)
- In most interesting cases it is definable using both prediction and compression
- Every set in assigned an effective dimension
Let us move to definitions ...
Our notation for Cantor space

- For $\Sigma$ a finite alphabet, $\Sigma^*$ is the set of finite sequences over $\Sigma$ ($\{0, 1\}^*$)
- $\{0, 1\}^\infty$ is the set of infinite binary sequences
- For $x \in \{0, 1\}^\infty$, $x \upharpoonright n$ the the length $n$ finite prefix of $x$

- In Computational Complexity we will identify a problem/language $A \subseteq \{0, 1\}^*$ with its characteristic (infinite) sequence $\chi_A \in \{0, 1\}^\infty$
- Otherwise we may be interested in the real number in $[0, 1]$ represented by each $x \in \{0, 1\}^\infty$ (the number with binary representation $0.x$) the choice of alphabet can be relevant
Lutz gambling characterization of dimension in Cantor space

- For \( s \in [0, \infty) \), an \( s \)-supergale is a function \( d : \{0, 1\}^* \to [0, \infty) \) such that \( w \in \{0, 1\}^* \)

\[
d(w) \geq \frac{d(w0) + d(w1)}{2^s}
\]

- The success set of an \( s \)-supergale \( d \) is

\[
S^\infty[d] = \left\{ x \in \{0, 1\}^\infty \bigg| \lim_{n} \sup d(x \upharpoonright n) = \infty \right\}
\]

**Theorem**

*For every \( A \subseteq \{0, 1\}^\infty \),

\[
\dim_H(A) = \inf \{ s \mid \text{there is an } s \text{-supergale } d \text{ such that } A \subseteq S^\infty[d] \}\]
Variants

- Use $\lim \inf$ in the success definition:
  $$S_{\text{str}}^\infty[d] = \{ x \in \{0, 1\}^\infty \mid \lim \inf_n d(x \mid n) = \infty \}$$
  to characterize packing dimension

- Use martingale growth rates in the place of gales

- gales or supergales
Constructive dimension in Cantor space

The constructive dimension of $A$ is

$$\text{cdim}(A) = \inf \left\{ s \mid \text{there is a constructive } s\text{-supergale } d \text{ such that } A \subseteq S^\infty[d] \right\}$$

Constructive means lower semi-computable, that is $d$ is constructive if there is an exactly computable function $\hat{d} : \Sigma^* \times \mathbb{N} \rightarrow \mathbb{Q}$ with the following two properties.

- For all $w \in \Sigma^*$ and $t \in \mathbb{N}$, $\hat{d}(w, t) \leq \hat{d}(w, t + 1) < d(w)$.
- For all $w \in \Sigma^*$, $\lim_{t \to \infty} \hat{d}(w, t) = d$. 
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Let $\Delta$ be a class of functions (e.g., polynomial time computable, polynomial space computable)

The $\Delta$-dimension of $A$ is

$$\dim_{\Delta}(A) = \inf \left\{ s \mid \text{there is an } s\text{-supergale } d \in \Delta \text{ such that } A \subseteq S^\infty[d] \right\}$$

- Choosing different $\Delta$ we restrict gales to different classes of computable strategies
- With gales computable by a finite automata we get $\dim_{\text{FS}}$
- $\dim_{p}$ corresponds to computable in polynomial time
- $\dim_{\text{pspace}}$ means polynomial space computable gales

Each of this effective dimensions is “the right one” for a set of sequences (complexity class)
Complexity classes

Each r-b dimension is the right one for a complexity class

- $E = \text{DTIME}(2^{O(n)})$, we have $\dim_p(E) = 1$
- $\text{EXP} = \text{DTIME}(2^{n^{O(1)}})$, $p_2$ is $2^{\text{polylog}}$ time computable, we have $\dim_{p_2}(\text{EXP}) = 1$
- $\text{ESPACE} = \text{DSPACE}(2^{O(n)})$, we have $\dim_{\text{pspace}}(\text{ESPACE}) = 1$
- $\text{EXPSPACE} = \text{DSPACE}(2^{n^{O(1)}})$, $p_2$ space is $2^{\text{polylog}}$ space computable, we have $\dim_{p_2\text{space}}(\text{EXPSPACE}) = 1$
- $\dim_{FS}(\mathbb{Q}) = 1$

Sometimes we denote $\dim_p(X \cap E)$ as “dimension in $E$ of $X$”, etc.
Some properties or resource-bounded dimension

- \( \dim_{\Delta}(X) \) is defined for every set \( X \)
- \( X \subseteq Y \) implies \( \dim_{\Delta}(X) \leq \dim_{\Delta}(Y) \)
- \( \dim_{\Delta}(\bigcup_i X_i) = \sup_i \dim_{\Delta}(X_i) \) for “suitable” effective unions

where \( \dim_{\Delta} \) is any of the effective dimensions
Uses of effective dimension in complexity

- Abundance proofs
- Probabilistic method
- New hypothesis, new concepts
The class of sets that (polynomial-time) reduce to a nondense set has p-dimension 0 in Exponential time (E)

E has p-dimension 1

Most sets in Exponential time do not reduce to a nondense set
Abundance result in detail

How dense are hard sets for exponential time?

- The most common notions of polynomial time reductions are many-one $\leq^p_m$ and Turing $\leq^p_T$
- In between $\leq^p_m$ and $\leq^p_T$ is a wide variety of polynomial-time reductions of different strengths
- Reductions are often used to prove hardness for a complexity class, we will look at $E$ and $EXP$

$$DENSE = \left\{ L \mid \exists \epsilon \forall n |L^{\leq n}| > 2^{n\epsilon} \right\}$$

- All known hard problems for $E$ and $EXP$ are dense
- Is every hard set dense?
Density of hard sets

Known:

- (Watanabe 1987) Every hard set for E under the $\leq_{\log -tt}^p$ reductions is dense
- (Lutz Mayordomo 1994) Every hard set for E under the $\leq_{n^\alpha -tt}^p (\alpha < 1/2)$ reductions is dense
- (Fu 1995, Lutz Zhao 2000) Every hard set for E under the $\leq_{n^\alpha -T}^p (\alpha < 1/2)$ reductions is dense. Every hard set for EXP under the $\leq_{n^\alpha -T}^p (\alpha < 1)$ reductions is dense

Curious contrast E, EXP ...
(Hitchcock 2005, Harkins Hitchcock 2011) improved all previous results by showing the following result:

**Theorem**

The p-dimension of sets that reduce to nondense sets (under $\leq_{n^\alpha-T}$ ($\alpha < 1$) reduction) is 0

Their proof is quite involved, including:
- the online mistake-bound model of learning
- reduction to learnable concepts
- the set of reducible to learnable concepts has p-dimension 0
- sets that reduce to nondense are reducible to learnable classes (monotone disjunctions with few literals)

**Abundance result** ($\dim_p(E) = 1$) Most sets in E do not reduce to nondense sets

**Existence result** (probabilistic method) There is a set in E that does not $\leq_{n^\alpha-T}$-reduce ($\alpha < 1$) to nondense sets

**Consequence:** All $\leq_{n^\alpha-T}$-hard sets for E are dense
A taste of effective dimension: Probabilistic method

- $\dim_p(\text{absly} \text{ – normal}) = 1$ (The set of absolutely normal numbers have polynomial-time dimension 1)

- A real number $\alpha$ is normal in base $b$ (Borel 1909) if the base $b$ representation of $\alpha$ for every finite sequence $w$ of base $b$ digits the asymptotic, empirical frequency of $w$ in the base-$b$ expansion of $\alpha$ is $b^{|w|}$

- Absolutely-normal number means normal in every base

- The result implies an efficient way of constructing an absolutely normal real number (constructive probabilistic method)

I will get back to this in my next lecture
A taste of effective dimension: new hypothesis

- It is not known whether all NP-hard sets are dense
- If \( \dim_p(NP) > 0 \) then all \( \leq_{n^\alpha - T}^p \)-hard sets for NP are dense
A taste of effective dimension: new hypothesis

- MAX3SAT is the problem of computing the number of satisfied clauses in a 3SAT formula.
- If \( \dim_p(NP) > 0 \) then MAX3SAT is hard to approximate (effective approximation algorithms have performance ration less than 7/8 on a dense set of instances).
A taste of effective dimension: new hypothesis

- **BPP** is the class of problems solvable in bounded error probabilistic polynomial time
- **Zero-One law**: \( \dim_{p_2}(\text{BPP}) = 0 \) or \( \text{BPP} = \text{EXP} \)
Let $g: \mathbb{N} \times [0, \infty) \to [0, \infty)$ be a scale function (a family of gauge functions).

Usual Hausdorff dimension corresponds to the scale $g(m, s) = sm$.

For $s \in [0, \infty)$, an $g$-$s$-supergale is a function $d: \{0, 1\}^* \to [0, \infty)$ such that $w \in \{0, 1\}^*$

$$d(w) \geq \frac{d(w0) + d(w1)}{2g(|w|+1,s)-g(|w|,s)}$$

The success set of an $g$-$s$-supergale $d$ is

$$S^\infty[d] = \left\{ x \in \{0, 1\}^\infty \left| \limsup_{n} d(x \upharpoonright n) = \infty \right. \right\}$$

Theorem

For every $A \subseteq \{0, 1\}^\infty$,

$$\dim_H^g(A) = \inf \{ s \mid \text{there is a } g$-$s$-$\text{supergale } d \text{ such that } A \subseteq S^\infty[d] \}$$
Resource-bounded dimension: changing the scale

- Related to the classical concept of exact or general dimension
- We consider different scales $g$ for which $\dim_g^p(E) = 1$, $\dim_g^{\text{pspace}}(\text{ESPACE}) = 1$
- For certain scales $g, g'$ it holds that that

\[
\dim_{g}^{\text{pspace}}(\text{SIZE}(2^{\alpha n})) = \alpha
\]

\[
\dim_{g'}^{\text{pspace}}(\text{SIZE}(2^{n^\alpha})) = \alpha
\]
A taste of effective dimension: small span theorems

- Small span theorems: given a reduction, either the upper or the lower span is small
- For a language $A$ and a reduction $r$, the upper spam is
  \[ P_{r}^{-1}(A) = \{ B \mid A \leq_{r} B \} \]
- For a language $A$ and a reduction $r$, the lower spam is
  \[ P_{r}(A) = \{ B \mid B \leq_{r} A \} \]

Theorem

For every $A$ in $E$, either

\[ \dim^{g}_{p}(P_{m}(A)) = 0 \]

or

\[ \dim^{g}_{p}(P_{m}^{-1}(A)) = 0 \]
Open questions on resource-bounded dimension

- What is the $p$-dimension of NP?
- Is it possible that $0 < \dim_p(\text{NP}) < 1$?
A problem $X$ is complete for a class $C$ if every $Z \in C$ can be reduced to $X$.

A problem $X$ is *partially complete* for a class $C$ if the set of $Z \in C$ that can be reduced to $X$ has nonzero dimension in $C$.

**OPEN:**
- Examples of natural partially complete problems
- Is Graph Isomorphism partially complete for EXP?
- Are partially complete the same for E and EXP?
In certain ways positive dimension can substitute Martin-Löf randomness.

It was known that for each Martin-Löf random $x$, $\text{BPP} \subseteq \text{P}^x$ (in fact for much lower resource-bounded randomness).

Can I have $\text{P}^A = \text{BPP}$ when $\text{dim}_p(A) > 0$?
Main references

References for resource-bounded dimension

Computational Complexity results in this lecture


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