Reverse Mathematics of Model Theory and First-Order Principles
Denis R. Hirschfeldt — University of Chicago
IMS, National University of Singapore, September 2017

Aspects of Computation, in Celebration of Rod Downey
**Induction and bounding**

$I\Sigma^0_n$ is induction for $\Sigma^0_n$ formulas.

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$\textbf{B}\Sigma_n^0$ is bounding for $\Sigma_n^0$ formulas:

$$\forall i < n \exists x \varphi(i, x) \rightarrow \exists b \forall i < n \exists x < b \varphi(i, x).$$
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$B\Sigma^0_n$ is bounding for $\Sigma^0_n$ formulas:

$$\forall i < n \exists x \varphi(i, x) \rightarrow \exists b \forall i < n \exists x < b \varphi(i, x).$$

Over $\text{RCA}_0$, $B\Sigma^0_n$ is strictly between $I\Sigma^0_{n-1}$ and $I\Sigma^0_n$.

**Thm (Slaman).** $B\Sigma^0_n$ is equivalent to $I\Delta^0_n$.

In particular, $\text{RCA}_0 \not\models B\Sigma^0_2$. 
Theories, structures, and trees

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$\mathcal{M}$ will denote a countable structure.
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RCA$_0$ proves that every $T$ has a model.

But what about models with special properties?

In particular ones determined by their type spectra.
\( \mathcal{M} \) is **atomic** if all types it realizes are principal.
The Atomic Model Theorem

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\( \text{RCA}_0 \) proves that if \( T \) has an atomic model then it is atomic.
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RCA$_0$ proves that if $T$ has an atomic model then it is atomic.

**Atomic Model Theorem (AMT).** If $T$ is atomic then it has an atomic model.
Thm (Goncharov and Nurtazin; Harrington). An atomic $T$ has a decidable atomic model iff there is a computable listing of the principal types of $T$. 

By considering trees of types and coding trees into theories, AMT can be restated as: If $V$ is a tree with isolated paths dense, then there is a listing of the isolated paths of $V$. Well, that is sort of true.
The combinatorial content of AMT

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The computability-theoretic content of AMT

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**Thm (Csima, Hirschfeldt, Knight, and Soare).** Let $X \leq_T \emptyset'$. Then $X$ is nonlow$_2$ iff every decidable atomic theory has an $X$-decidable atomic model.

**Cor.** $\text{WKL}_0 \nvdash \text{AMT}$. 
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**Thm (Csima, Hirschfeldt, Knight, and Soare; Conidis)** Every decidable atomic theory has an $X$-decidable atomic model iff no $\Delta^0_2$ function dominates every $X$-computable function.
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**Cor.** $\text{WKL}_0 \nvDash \text{AMT}$. 
The reverse-mathematical content of AMT

**ADS**: Every infinite linear order has an infinite ascending or descending sequence.

A linear order is **stable** if every element has either finitely many predecessors or finitely many successors.

**SADS**: Every stable infinite linear order has an infinite ascending or descending sequence.

**Thm (Hirschfeldt and Shore)**. \( RT^2 \leftrightarrow ADS \leftrightarrow SADS. \)
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**Thm (Hirschfeldt and Shore)**. \( \mathsf{RT}_2^2 \leftrightarrow \mathsf{ADS} \leftrightarrow \mathsf{SADS} \).

**Thm (Hirschfeldt, Shore, and Slaman)**. \( \mathsf{SADS} \leftrightarrow \mathsf{AMT} \).

**Thm (Hirschfeldt, Shore, and Slaman)**. AMT is \( \Pi^1_1 \)-conservative over \( \Sigma^0_1 \), \( \mathcal{B} \Sigma^0_2 \), and \( \mathcal{I} \Sigma^0_2 \).
AMT and genericity

Being an atom of a decidable theory is a $\Pi^0_1$ property.
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$\Pi^0_1G$: For any uniformly $\Pi^0_1$ dense predicates $D_0, D_1, \ldots$ on $2^{<\omega}$, there is a $G \in 2^\omega$ that meets each $D_i$. 
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$\Pi^0_1G$ implies AMT.

Thm (Conidis). AMT implies $\Pi^0_1G$ over $I\Sigma^0_2$. 
AMT and genericity

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$\Pi^0_1 G$ implies AMT.

**Thm (Conidis).** AMT implies $\Pi^0_1 G$ over $I\Sigma^0_2$.

**Thm (Hirschfeldt, Shore, and Slaman).** $\Pi^0_1 G$ is $\Pi^1_1$-conservative over $I\Sigma^0_1$ and $I\Sigma^0_2$, but implies $I\Sigma^0_2$ over $B\Sigma^0_2$. 
First-Order Questions

NO MATTER HOW MANY MUST DIE, NO MATTER HOW MANY CITIES MUST BE DESTROYED... WE WILL, IN THE END, BE VICTORIOUS! REMEMBER THIS ALWAYS... AND GIVE UP YOUR BODIES, YOUR MINDS... YOUR VERY LIVES TO OUR ULTIMATE GOAL...

"HYDRA OVER ALL!"

HAIL HYDRA! IMMORTAL HYDRA!

WE SHALL NEVER BE DESTROYED!

CUT OFF ONE ARM, AND TWO MORE WILL TAKE ITS PLACE!

HYDRA, MOST SWIFT OF ALL INDESTRUCTIBLE ENEMIES! HYDRA, DEFEATED OF ALL THREATS TO DEMOCRACY AND WORLD PEACE...

AND IF YOU THINK THIS IS JUST ANY OLD HYDRA MEETING... READ ON, BROTHER, AND FIND OUT HOW WRONG YOU CAN BE...
The following is an attempt to capture the difference between AMT and $\Pi^0_1 G$:

$\Pi^0_1 GA$: For any uniformly $\Pi^0_1$ dense predicates $D_0, D_1, \ldots$ on $2^{<\omega}$, there are $g_s \in 2^\omega$ for $s \in \omega$ s.t.

$$\forall i \exists \sigma \in D_i \forall^\infty s (g_s \succ \sigma).$$
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**Thm (Hirschfeldt, Lange, and Shore).** $\Pi^0_1 \text{GA}$ is provable from $I\Sigma^0_2$ and equivalent to it over $B\Sigma^0_2$. 
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**Thm (Hirschfeldt, Lange, and Shore).** $\Pi^0_1 GA$ is provable from $I\Sigma^0_2$ and equivalent to it over $B\Sigma^0_2$.

**Open Question.** Does AMT + $\Pi^0_1 GA$ imply $\Pi^0_1 G$?
$\Pi^0_n G$ For any uniformly $\Pi^0_n$ dense predicates $D_0, D_1, \ldots$ on $2^{<\omega}$, there is a $G \in 2^\omega$ that meets each $D_i$. 

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\( \Pi^0_n G \) For any uniformly \( \Pi^0_n \) dense predicates \( D_0, D_1, \ldots \) on \( 2^{< \omega} \), there is a \( G \in 2^\omega \) that meets each \( D_i \).

\( \Pi^0_n GA \) For any uniformly \( \Pi^0_n \) dense predicates \( D_0, D_1, \ldots \) on \( 2^{< \omega} \), there are \( g_{s_0, \ldots, s_{n-1}} \in 2^\omega \) for \( s_0, \ldots, s_{n-1} \in \omega \) s.t.

\[
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**Thm (Hirschfeldt, Lange, and Shore).** \( \Pi^0_n GA \) is provable from \( I\Sigma^0_n \) and equivalent to it over \( B\Sigma^0_n \).
Let us be more precise about the restatement of AMT as: If $V$ is a tree with isolated paths dense, then there is a listing of the isolated paths of $V$. 

An atom of a tree $V$ is a node contained in exactly one infinite path of $V$. $V$ is atomic if every node of $V$ can be extended to an atom. $V$ is strongly atomic if for every finite sequence $\sigma_0, \ldots, \sigma_n \in V$, there is a finite sequence $\tau_0, \ldots, \tau_n$ of atoms of $V$ s.t. $\sigma_i \preceq \tau_i$. 
Atomic trees

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$V$ is **strongly atomic** if for every finite sequence $\sigma_0, \ldots, \sigma_n \in V$, there is a finite sequence $\tau_0, \ldots, \tau_n$ of atoms of $V$ s.t. $\sigma_i \preceq \tau_i$. 
Atomic and strongly atomic coincide for theories:

If a theory $T$ is atomic and $\varphi_0(\vec{x}_0), \ldots, \varphi_n(\vec{x}_n)$ are consistent with $T$, then we can extend them all simultaneously to atoms by extending $\varphi_0(\vec{y}_0) \land \cdots \land \varphi_n(\vec{y}_n)$, where the $\vec{y}_i$ are pairwise disjoint.
The Atomic Tree Theorem

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Thm (Hirschfeldt, Lange, and Shore). The statement that every atomic tree is strongly atomic is equivalent to $\mathsf{B}\Sigma^0_2$.

Open Question. Does $\mathsf{AMT} (+ \Pi^0_1 \mathsf{GA})$ imply $\mathsf{ATT}$?
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**AMT** is equivalent to: If $V$ is a strongly atomic tree then there is a listing of the isolated paths of $V$.
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AMT is equivalent to: If $V$ is a strongly atomic tree then there is a listing of the isolated paths of $V$.

**ATT**: If $V$ is an atomic tree then there is a listing of the isolated paths of $V$.

$\Pi^0_1 G$ implies ATT, which in turn implies AMT.

Open Question. Does AMT ($+ \Pi^0_1 GA$) imply ATT?
Further variants

$\Pi^0_1^{GA}$ is a miniaturization of $\Pi^0_1^{G}$. Here is a miniaturization of ATT:

**FATT**: If $V$ is an atomic tree then for every sequence $\sigma_0,\ldots,\sigma_n \in V$, there is a sequence $P_0,\ldots,P_n$ of isolated paths of $V$ s.t. $\sigma_i \prec P_i$. 
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FATT is implied by $B\Sigma^0_2$ and by ATT (and so does not imply $B\Sigma^0_2$).

**Thm (Hirschfeldt, Lange, and Shore)**. FATT does not hold in RCA$_0$. 

Open Question. Does $\Pi^0_1 \text{GA}$ imply FATT? 

ATT$^{-}$: Let $V$ be a tree s.t. for every sequence $\sigma_0, \ldots, \sigma_n \in V$, there is a sequence $P_0, \ldots, P_n$ of isolated paths of $V$ s.t. $\sigma_i \prec P_i$. Then there is a listing of the isolated paths of $V$.

ATT implies ATT$^{-}$, which in turn implies AMT. 

Open Question. Does AMT ($+\Pi^0_1 \text{GA}$) imply ATT$^{-}$? 
Π₁⁰GA is a miniaturization of Π₁⁰G. Here is a miniaturization of ATT:

**FATT**: If $V$ is an atomic tree then for every sequence $σ_0, \ldots, σ_n ∈ V$, there is a sequence $P_0, \ldots, P_n$ of isolated paths of $V$ s.t. $σ_i ≺ P_i$.

FATT is implied by BΣ₂ and by ATT (and so does not imply BΣ₂).

**Thm (Hirschfeldt, Lange, and Shore)**. FATT does not hold in RCA₀.

**Open Question**. Does Π₁⁰GA imply FATT?
Further variants

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Further variants

$\Pi^0_1 \text{GA}$ is a miniaturization of $\Pi^0_1 \Theta$. Here is a miniaturization of ATT:

**FATT:** If $V$ is an atomic tree then for every sequence $\sigma_0, \ldots, \sigma_n \in V$, there is a sequence $P_0, \ldots, P_n$ of isolated paths of $V$ s.t. $\sigma_i \prec P_i$.

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ATT implies ATT$^-$, which in turn implies AMT.

**Open Question.** Does AMT (+ $\Pi^0_1 \text{GA}$) imply ATT$^-$?
Homogeneous Models

Cartoon by George du Maurier (Punch, 1895)

TRUE HUMILITY

Right Reverend Host: “I’m afraid you’ve got a bad egg, Mr. Jones!”
The Curate: “Oh no, my lord, I assure you! Parts of it are excellent!”
Classically, the following definitions are equivalent:

1. $\mathcal{M}$ is homogeneous if for all $\vec{a} \equiv \vec{b} \in \mathcal{M}$, $(\mathcal{M}, \vec{a}) \cong (\mathcal{M}, \vec{b})$.

2. $\mathcal{M}$ is homogeneous if for all $\vec{a} \equiv \vec{b} \in \mathcal{M}$ and all $c \in \mathcal{M}$, there is a $d \in \mathcal{M}$ s.t. $\vec{a}c \equiv \vec{b}d$. 
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The implication from 2 to 1 is equivalent to ACA$_0$. 

Homogeneous models with the same type spectra are isomorphic. This statement is equivalent to ACA$_0$. 

Every $\mathcal{T}$ has a homogeneous model. Thm (Macintyre and Marker; Csima, Harizanov, Hirschfeldt, and Soare; Lange; Belanger). This statement is equivalent to WKL$_0$. 

Uniqueness and existence of homogeneous models
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The implication from 2 to 1 is equivalent to $\text{ACA}_0$.

Homogeneous models with the same type spectra are isomorphic.

This statement is equivalent to $\text{ACA}_0$. 
Classically, the following definitions are equivalent:

1. \( M \) is homogeneous if for all \( \vec{a} \equiv \vec{b} \in M \), \( (M, \vec{a}) \cong (M, \vec{b}) \).

2. \( M \) is homogeneous if for all \( \vec{a} \equiv \vec{b} \in M \) and all \( c \in M \), there is a \( d \in M \) s.t. \( \vec{a}c \equiv \vec{b}d \).

The implication from 2 to 1 is equivalent to ACA\(_0\).

Homogeneous models with the same type spectra are isomorphic.

This statement is equivalent to ACA\(_0\).

Every \( T \) has a homogeneous model.

Thm (Macintyre and Marker; Csima, Harizanov, Hirschfeldt, and Soare; Lange; Belanger). This statement is equivalent to WKL\(_0\).
An alternate definition

Recall:

1. $\mathcal{M}$ is homogeneous if for all $\vec{a} \equiv \vec{b} \in \mathcal{M}$, $(\mathcal{M}, \vec{a}) \simeq (\mathcal{M}, \vec{b})$.

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The following definition is sometimes reverse-mathematically better behaved:

3. $\mathcal{M}$ is homogeneous if for all $\vec{a}_0 \equiv \vec{b}_0, \ldots, \vec{a}_n \equiv \vec{b}_n \in \mathcal{M}$ and $\vec{c}_0, \ldots, \vec{c}_n \in \mathcal{M}$, there are $\vec{d}_0, \ldots, \vec{d}_n \in \mathcal{M}$ s.t. $\vec{a}_i \vec{c}_i \equiv \vec{b}_i \vec{d}_i$ for $i \leq n$. 

Thm. The implication from 2 to 3 is equivalent to $\text{I} \Sigma_0^2$. 
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The Homogeneous Model Theorem

**HMT (Goncharov; Peretyat’kin).** Let $S$ be a countable set of types of $T$. There is a countable homogeneous model with type spectrum $S$ iff $S$ satisfies the following closure conditions:

- $T \in S$
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First-order issues

Without $\text{I} \Sigma_2^0$, the equivalence between AMT and HMT is sensitive to the choice of definitions of homogeneity and of closure under type amalgamation.

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It is an open question whether these are equivalent to each other or to $\Pi^0_1 \text{GA}$.
Saturated Models

Figure 1:
Map showing the location of Starbucks coffee houses in and around downtown San Francisco, California, USA. In addition to these coffee shops, there are many other chain and independent stores in the area.
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**Cor (Harris).** If $d$ is either high or PA then $d$ is saturated-bounding.

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