Lowness notions in the C.E. Sets

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The End

Thanks!

Happy Birthday Rod!

Main New Result

Theorem

Let $\mathbf{e}$ *be any Turing degree such that* $\mathbf{e}$ *is computably enumerable in* $\mathbf{0}'$*. Then*

- *There is a (noncompupable) c.e. set* C *such that* $C' \equiv_T \mathbf{e}$ *(Sack Jump Inversion).*
- *If* A *is the Dekker deficiency set of* C *then* $\overline{A}$ *is semilow*$_2$*.*

Dekker deficiency set

Let f be the computable $1-1$ function whose range is C (given to us by the above construction). The Dekker deficiency set is

$$A = \{s : (\exists t > s)[f(t) < f(s)]\}.$$

Lemma

A is c.e., of degree C, and hsimple (so $\overline{A}$ is hyperimmune).

Deficiency sets and hhsimple

Theorem (Shoenfield 1976)

If a deficiency set A has a hhsimple superset H then A is low$_2$.

Corollary

There is a nonhigh nonlow$_2$ c.e. set A such that A does not have a maximal superset and $\overline{A}$ is semilow$_2$.

Definition

M is maximal iff, for all e, either $W_e \subseteq^* M$ or $M \cup W_e \subseteq^* \omega$.

Sets with maximal supersets

Theorem (Lachlan 1968)

If A (is infinite c.e.) and low$_2$ then A has a maximal superset, M.

- Since A is nonhigh, A has a true stage enumeration. An enumeration $\{A_s | s \in \omega\}$ such that for infinite many s, $a_s = a_s^s$, where $\overline{A}_s = \{a_0^s < a_1^s \dots\}$ and $\overline{A}_s = \{a_0 < a_1 \dots\}$. So, at true stage, $a_s^s = a^s$. (Access to $\overline{A}$.)
- Since A is low$_2$, the set of indexes e such that $\{x | x \in W_{e,s}, x \notin A_s,$ and s is a true stage$\}$ is infinite is computable in $\mathbf{0}''$. (Information.)

An imperfect stream of balls outside of A

Using $\mathbf{0}''$ we can ask if $\{x | x \in W_s, x \notin M_s, \text{ and } s \text{ is a true stage}\}$ is infinite. If yes, we are guaranteed for all k there will be stage s such that there at least k balls x where $x \in W_s, x \notin M_s$ and s is a true stage so these x are not in A. But we have no way to bound how long it will take for the $(k+1)$th ball to stabilize.

Using this imperfect stream

Infinitely often when we have verification that the set $\{x | x \in W_s, x \notin M_s, \text{ and } s \text{ is a true stage}\}$ is infinite, we can dump the balls out in W into M.

We can safety take exactly one action on this stream. We cannot take half and put them into M_1 and the other half into M_2 and hope both these c.e. sets are disjoint and infinite outside A. We cannot divide this imperfect stream into two imperfect streams.

Soare's Result

Definition

The outside of A is denoted $\mathcal{L}(A)$ which is the structure $\{W_e \cup A | e \in \omega\}$ under inclusion. $\mathcal{E}$ is the structure $\{W_e | e \in \omega\}$ under inclusion.

Note that if $A = \emptyset$ then $\mathcal{L}(A) = \mathcal{E}$.

Theorem (Soare)

If A is low then $\mathcal{L}(A)$ and $\mathcal{E}$ are isomorphic.

Question

For which A are $\mathcal{L}(A)$ and $\mathcal{E}$ are isomorphic? It was conjectured that A can be any low$_2$ set.

This question is about lowness notions. If A realizes one of our lowness notions then we want that $\mathcal{L}(A)$ and $\mathcal{E}$ are isomorphic. Since maximal set exists, A must have a maximal superset.

Main New Result, again

Corollary

There is a nonhigh nonlow$_2$ c.e. set A such that $\mathcal{L}(A)$ is not isomorphic to $\mathcal{E}$ and $\overline{A}$ is semilow$_2$.

Such an A has a true stages enumeration.

(Soare's) Information lowness or Semilow$_2$

We want infinitely many balls outside of A.

Definition

B is *semilow$_2$* iff $\{e | W_e \cap B \text{ is infinite}\} \leq_T \mathbf{0}''$.

If A is low$_2$ then $\overline{A}$ is semilow$_2$. Outside of low, low$_2$ and nonhigh are our lowness notions are not properties of Turing degrees.

(Soare's) access to $\overline{A}$

Definition

$\overline{A}$ is *semilow* iff $\{e | W_e \cap \overline{A} \neq \emptyset\} \leq_T \mathbf{0}'$.

This a $\Sigma_1^{\overline{A}}$ question. If A is low then this question is Δ_2^0.

Use semilowness of $\overline{A}$ and the limit lemma to uniformly split ω (or any W_e we know is infinite outside A) into the disjoint union of *finite* sets F_i such that, for all i, $F_i \cap \overline{A}$ is nonempty. At stage s if our approximation of $\mathbf{0}'$ says that the set $(\omega - \bigsqcup_{i<e} F_e) \cap \overline{A}$ is nonempty but $F_e \cap \overline{A}$ is empty, put the element x of ω which enters at stage s into F_e (for the least such e), otherwise x goes into F_s.

The F_i provide finite access to the outside of A. We can put half into M_1 and the other half into M_2. We can split an infinite stream of balls outside A into 2.

For our imperfect streams we have no finite access nor can we split streams.

Semilow

Theorem (Soare)

If $\overline{A}$ is semilow then $\mathcal{L}(A)$ and $\mathcal{E}$ are (effectively) isomorphic.

Semilow$_{1.5}$

Definition (Maass)

B is *semilow*$_{1.5}$ iff
$\{e | W_e \cap B \text{ is infinite}\} \leq_m \{e | W_e \text{ is infinite}\} = INF$.

Stronger than semilow$_2$, weaker than semilow.

Theorem (Maass)

If $\overline{A}$ is semilow$_{1.5}$ then $\mathcal{L}(A)$ and $\mathcal{E}$ are isomorphic.

OSP

Definition

Lets assume that W is infinite outside of A. A *sieve* for W over A is an uniform collection of pairwise disjoint c.e. sets, $\{F_i | i \in \omega\}$, such that their union is W and, for all i, $F_i \cap \overline{A}$ is finite but nonempty.

A sieve witnesses that A is not hhsimple.

Lemma

A has osp iff, for all e, a sieve for W_e over A can be found uniformly.

Lemma (Maass)

If $\overline{A}$ is semilow$_{1.5}$ then A has osp.

All streams of balls outside A can be split into 2 such streams uniformly when A has osp.

End of the line

Theorem (Classic Cholak)

If A has osp and $\overline{A}$ is semilow$_2$ then $\mathcal{L}(A)$ and $\mathcal{E}$ are isomorphic.

Corollary (Main New Result)

There is an A with a true stages enumerations, $\overline{A}$ is semilow$_2$ and $\mathcal{L}(A)$ is not isomorphic to $\mathcal{E}$.

True stages cannot replace osp.

The low_2 question

There are low_2 sets without osp.

Question

If A is low_2 are $\mathcal{L}(A)$ and $\mathcal{E}$ are isomorphic?

Likely false. Is there a definable property, P such that $P(\emptyset)$ holds but fails for $\overline{A}$? A definable version of the failure to split a stream into two.
Perhaps true? The modern automorphism method needs the finite access and the ability to split a stream into 2. So forced to use Soare's old effective automorphism method and chip sets.

Question

Do all low_2 sets have atomless hhsimple superset?