Weakly Ultrahomogeneous Structures

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Outline

- New notion of *weakly ultrahomogeneous*
- Characterization of weakly homogeneous structures
  - linear orderings,
  - *n*-equivalence structures
  - injection structures
  - trees
- Effective categoricity of such structures
- Joint work with Francis Adams

Finitely Generated Substructures

Definition (R. Fraisse)

The age of a structure $\mathcal{A}$ is the family of finite substructures of $\mathcal{A}$ (up to isomorphism).

Example

The age of $\mathbb{Q}, <)$ *is the set of finite linear orders.*

Example

The age of $\mathcal{P}(\mathbb{N}), \cup, \cap, ^c)$ *is the set of finite Boolean algebras.*

Properties of Ages

If K is the age of a structure $\mathcal{D}$, then K satisfies:

1. **Hereditary Property** (HP): If $A \in K$, then any finitely generated substructure of $\mathcal{A}$ is in K.
2. **Joint Embedding Property** (JEP): If $\mathcal{A}, \mathcal{B} \in K$, then there is $\mathcal{C} \in K$ such that $\mathcal{A}, \mathcal{B}$ both embed in $\mathcal{C}$.

Fraisse Limits

Definition (R. Fraisse)

A structure is ultrahomogeneous if any isomorphism between finitely generated substructures extends to an automorphism of the whole structure.

Fraisse defined an amalgamation property (AP) and proved.

Theorem (Fraisse)

If K is an age with the AP, then there exists a countable ultrahomogeneous $\mathcal{A}$ with age K. $\mathcal{A}$ is the Fraisse limit of K.

Example

1. $(Q, <)$, *the countable dense linear order, is the Fraisse Limit of the finite linear orders.*
2. *The countable atomless boolean algebra is the limit of the finite Boolean algebras*

Extension Property

The key to an ultrahomogeneous structure is the following:

Given an isomorphism H between two finitely generated substructures $< a_1, \ldots, a_k >$ and $< b_1, \ldots, b_k >$, and an element a, there exists an element b and an extension of H mapping $< a_1, \ldots, a_k, a >$ to $< b_1, \ldots, b_k, b >$.

This can also be applied to mappings between two isomorphic ultrahomogeneous structures.

The classic example is the dense linear ordering without end points $(\mathbb{Q}, <)$

Other Countable Ultrahomogeneous Structures

The Random graph is the limit of the finite graphs.

Equivalence Structures with every orbit of the same size.

Injection Structures (A, f) with no orbits of type ω.

Weakly Ultrahomogeneous Structures

Definition

A structure $\mathcal{A}$ is weakly ultrahomogeneous if there exists a finite set $\{a_1, a_2, \ldots, a_n\} \subseteq A$ such that for all tuples $\vec{x}, \vec{y}$ from A with $\langle \vec{a}, \vec{x} \rangle \cong \langle \vec{a}, \vec{y} \rangle$ where each a_i is fixed, this isomorphism of substructures extends to an automorphism of $\mathcal{A}$. Call such a set $\{a_1, a_2, \ldots, a_n\}$ an exceptional set of $\mathcal{A}$.

Alternatively, $\mathcal{A}$ is weakly ultrahomogeneous if there is a finite set $a_1, \ldots, a_n$ such that $(\mathcal{A}, a_1, \ldots, a_n)$ is ultrahomogeneous in the extended language with constants for the a_i.

Effective Categoricity

Definition

- *A computable structure $\mathcal{A}$ is Δ^0_α categorical if every computable structure isomorphic to $\mathcal{A}$ is Δ^0_α isomorphic to $\mathcal{A}$; $\mathcal{A}$ is computably categorical if every computable structure isomorphic to $\mathcal{A}$ is computably isomorphic to $\mathcal{A}$.*
- *An arbitrary structure $\mathcal{A}$ is* **relatively** Δ^0_α *categorical if for any structure $\mathcal{B}$ isomorphic to $\mathcal{A}$, there is an isomorphism which is Δ^0_α relative to both $\mathcal{A}$ and $\mathcal{B}$.*

Main Computability Theorem

Theorem

Every computable weakly ultrahomogeneous structure is Δ^0_2-categorical.

Sketch: Checking if two finitely generated substructures are isomorphic is Π^0_1. Then a back-and-forth construction using oracle $\mathbf{0}'$ builds an isomorphism between given structures.

Locally Finite Structures

$\mathcal{A}$ is *locally finite* if every finite set generates a finite substructure.

Corollary

Any relational, or more generally, any locally finite, computable weakly ultrahomogeneous structure is computably categorical.

The converse is false; a single $\mathbb{Z}$-orbit is computably categorical.

Classification of Linear Orders

Theorem

For a countable linear order $\mathcal{A}$, the following are equivalent:

1. $\mathcal{A}$ *is weakly ultrahomogeneous.*
2. $\mathcal{A}$ *has finitely many successivities.*
3. $\mathcal{A} = L_0 + \mathbb{Q} + L_1 + \mathbb{Q} + \ldots + \mathbb{Q} + L_n$ *where the* L_i *are finite chains,* L_0, L_n *are possibly empty and* $|L_i| \geq 2$ *for* $1 \leq i \leq n-1$.

Theorem

Every weakly ultrahomogeneous linear order has a computable copy.

Effective Categoricity

Theorem (Remmel 1981)

A computable linear ordering is computably categorical iff it has finitely many successivities.

Theorem

A computable linear order is weakly ultrahomogeneous iff it is computably categorical.

Complexity of Being Weakly Ultrahomogeneous

- $\mathcal{A}_e = (\omega, <_e)$ where ϕ_e is the e'th partial recursive function, when ϕ_e is the characteristic function of a l.o.
- $LIN = \{e : \mathcal{A}_e \text{ is a linear ordering}\}$
- $UHL = \{e : \mathcal{A}_e \text{ is an ultrahomogeneous linear order}\}$
- $WUL = \{e \in LIN : \mathcal{A}_e \text{ is weakly ultrahomogeneous }\}$

Theorem

(a) *LIN and UHL are* Π^0_2 *complete.*

(b) *WUL is* Σ^0_3 *complete.*

In fact, *UHL* and *WUL* are complete relative to *LIN*.

Injection structures and Characters

An injection structure $\mathcal{A} = (A, f)$ consists of a set with a 1-to-1 function f.

The *orbit* of a is $\{x : (\exists n) f^{(n)}(x) = a \ \vee \ f^{(n)}(a) = x\}$.

Orbits may have type ω, type $\mathbb{Z}$, or finite type k.

The character K of $\mathcal{A}$ is
$\{(n, k) : \mathcal{A}$ has at least n orbits of type $k\}$

The character of a computable injection structure is a c.e. set.

Classification of Injection Structures

Proposition

Let $\mathcal{A}$ be an injection structure.

(a) *$\mathcal{A}$ is ultrahomogeneous iff it has no ω-orbits.*

(b) *$\mathcal{A}$ is weakly ultrahomogeneous iff it has finitely many ω-orbits.*

Effective Categoricity

By results of Cenzer, Harizanov, and Remmel

- A computable injection structure is computably categorical iff it has finitely many infinite orbits
- Such a structure is Δ_2^0-categorical iff it has finitely many ω-orbits or finitely many $\mathbb{Z}$-orbits.
- So for computable injection structures, computable categoricity implies weak ultrahomogeneity which implies Δ_2^0-categoricity.
- Neither implication can be reversed as witnessed by computable structures consisting of only infinitely many $\mathbb{Z}$-orbits, and of only infinitely many ω-orbits, respectively.

Structures with No Computable Copy

Let K be an arbitrary character and let $m, n \in \omega$:

1. There is an ultrahomogeneous injection structure $\mathcal{A}$ with character K and an arbitrary finite number of orbits of type $\mathbb{Z}$, which is relatively computably categorical.
2. There is a weakly ultrahomogeneous structure $\mathcal{A}$ with character K having m orbits of type $\mathbb{Z}$ and n of type ω, which is relatively computably categorical.
3. There is an ultrahomogeneous injection structure $\mathcal{B}$ with character K and with an infinite number of orbits of type $\mathbb{Z}$. Furthermore, $\mathcal{A}$ is relatively Δ^0_2 categorical.
4. There is a weakly ultrahomogeneous structure $\mathcal{B}$ with character K, infinitely many orbits of type $\mathbb{Z}$ and n orbits of type ω, which is relatively Δ^0_2 categorical.

Index Sets for Injection Structures

- $\mathcal{A}_e = (\omega, \phi_e)$ where ϕ_e when ϕ_e is an injection.
- $INJ = \{e : \mathcal{A}_e$ is an injection structure$\}$
- $UHI = \{e \in INJ : \mathcal{A}_e$ is ultrahomogeneous$\}$
- $WUI = \{e \in INJ : \mathcal{A}_e$ is weakly ultrahomogeneous$\}$

Theorem

(a) *INJ and UHI are Π^0_2 complete.*

(b) *WUI is Σ^0_3 complete.*

In fact, *UHI* and *WUI* are complete relative to *INJ*.

Classification of Equivalence Structures

Theorem

For a countable equivalence structure $\mathcal{A}$, $\mathcal{A}$ is weakly ultrahomogeneous iff all but finitely many equivalence classes of $\mathcal{A}$ are of the same size

Then by Cenzer, Harizanov, Calvert, Morozov (2005)

Corollary

A computable equivalence structure is weakly ultrahomogeneous iff it is computably categorical

Index Sets for Equivalence Structures

- $\mathcal{A}_e = (\omega, \mathbb{R}_e)$ when ϕ_e is the characteristic function of an equivalence relation R_e
- $EQ = \{e : \mathcal{A}_e$ is an equivalence structure$\}$
- $UHQ = \{e \in EQ : \mathcal{A}_e$ is ultrahomogeneous$\}$
- $WUQ = \{e \in INJ : \mathcal{A}_e$ is weakly ultrahomogeneous$\}$

Theorem

(a) *EQ and UHQ are Π^0_2 complete.*

(b) *WUQ is Σ^0_3 complete.*

In fact, UHQ and WUQ are complete relative to EQ.

Trees

Here a tree T is a subset of $\omega^{<\omega}$ which is closed under prefixes.

As a structure, T comes with a root ϵ (the empty string), a partial ordering ($\prec$, extension) and also
a predecessor function f, where $f(\sigma^\frown i) = \sigma$, and $f(\epsilon) = \epsilon$.

For $a \in T$, let $T(a) = \{x : x \prec a\}$, the tree below a

Let $T[a] = \{x : a^\frown x \in T\}$, the tree above a.

$(T, \prec)$ is a p.o. set such that each $T(a)$ is well-ordered by $\prec$.

Height and Rank

The height $ht_T(a)$ is the order type of $T(a)$.

The height $ht(T) = sup\{ht_t(a) : a \in T\}$

$ht(T)$ is always $\leq \omega$.

The rank $rk_T(x)$ is defined by recursion as

$$rk_T(x) = sup\{rk_T(y) + 1 : y \in T[x]\}$$

Then $rk(T) = rk_T(\epsilon)$.

$rk(T)$ can be any countable ordinal

Categoricity of Trees under $\prec$

Theorem (R. Miller)

If a computable $(T, \prec)$ *is computably categorical, then* T *has finite height.*

Lempp,McCoy,Miller,Solomon characterized the computably categorical trees of finite height under $\prec$.

Weakly Ultrahomogeneous Trees under $\prec$

Proposition

1. *$(T, \prec)$ is ultrahomogeneous iff $rk(T) \leq 1$*
2. *If $(T, \prec)$ is weakly ultrahomogeneous, then T has finite height.*
3. *$(T, \prec)$ is weakly ultrahomogeneous iff T has finitely many elements of rank ≥ 1.*

Thus every weakly ultrahomogeneous tree $(T, \prec)$ has a computable copy.

Every weakly u.h. computable tree $(T, \prec)$ is computably categorical, since trees are locally finite.

Trees with Predecessor

Let T have p.o. $\prec$ and predecessor function f.

Proposition

Let T have p.o. $\prec$ and predecessor function f.

1. *If $(T, \prec)$ is (weakly) u.h., then (T, f) is (weakly) u.h.*
2. *If $(T, \prec)$ is comp. cat., then (T, f) is comp. cat.*

The converse results do not hold.
(Take T with a single infinite path.)

Trees with predecessor are still locally finite, so any weakly u.h. tree must be relatively computably categorical.

Ultrahomogeneous Trees (T, f)

Theorem

(T, f) is ultrahomogeneous iff any two nodes of the same height have an equal number of successors.

Thus there are continuum many ultrahomogeneous trees.

Weakly Ultrahomogeneous Trees (T, f)

Definition

Let T be a tree with subtree S. $T_S[a]$ consists of nodes x such that either $x \preceq a$ or there is a successor b of a not in S such that $b \preceq x$.

Example

Let $T = \omega^{<\omega}$ and $S = \{x : x(0) > 1\}$.
Then $T_S[\epsilon] = \{\epsilon\} \cup \{x : x(0) \leq 1\}$.

Theorem

(T, f) is weakly ultrahomogeneous iff there is a finite subtree S of T such that $T_S[x]$ is u.h. for all $x \in S$.

Weakly Ultrahomogeneous Trees of Height ≤ 2

Proposition

A tree (T, f) of height ≤ 2 is weakly ultrahomogeneous iff all but finitely many nodes of height 1 have an equal number of successors.

Computably Categorical Trees not Weakly UH.

Example

Let T have infinitely many nodes of height 1 with exactly 2 successors and infinitely many with exactly 3 successors

Let each node of a pair of successors have exactly 4 successors and each node of a triple have exactly 1 successor.

This tree is computably categorical but not weakly uh in either presentation.

n-Equivalence Structures

Definition

An n-equivalence structure is $\mathcal{A} = (A, E_1, \ldots, E_n)$ where each E_i is an equivalence relation on A. $\mathcal{A}$ is nested if $i < j \Rightarrow xE_jy \rightarrow xE_iy$, i.e $E_j \subseteq E_i$ as subsets of $A \times A$.

Proposition

Let $\mathcal{A}$ be a n-equivalence structure, with $1 \leq n \leq \omega$. If $\mathcal{A}$ is ultrahomogeneous, then for $1 \leq i \leq n$, (A, E_i) is ultrahomogeneous.

Example of Comp. Cat. but not WUH structure

Example

Let E_1 *have equivalence classes* $\{0,1,2\},\{3,4,5\},\ldots$

Let E_2 *have equivalence classes* $\{0,1\},\{2,3\},\{4,5\},\ldots$

$\{1\}$ *and* $\{3\}$ *are isomorphic substructures, but this cannot be extended since* $card(\{0,1,2\}\cap\{0,1\})=2$ *but* $card(\{3,4,5\}\cap\{2,3\})=1$.

Trees and Equivalence Structures

Definition

For any n-equivalence structure $\mathcal{A} = (A, E_1, \ldots, E_n)$, let $E_0 = A \times A$, let E_{n+1} be equality, and define the tree $T_{\mathcal{A}}$ as follows. The universe of $T_{\mathcal{A}}$ is the set $\{[a]_i : a \in A, i = 1, \ldots, n\}$ and the partial ordering is inclusion. This means that for each a and $i \leq n$, $[a]_i$ is the predecessor of $[a]_{i+1}$.

A representation of $T_{\mathcal{A}}$ can be computed from $\mathcal{A}$ so that the map taking a to $[a]$ is also computable from $\mathcal{A}$.
This is due to Leah Marshall (Ph.D. thesis)

Categoricity of Trees and Equivalence Structures

Theorem (Marshall)

Let $\mathcal{A}$ be a computable n-equivalence structure and $T_{\mathcal{A}}$ its corresponding tree of finite height. Then the following are equivalent:

1. *$\mathcal{A}$ is computably categorical.*
2. *$\mathcal{A}$ is relatively computably categorical.*
3. *$(T_{\mathcal{A}}, \prec)$ is computably categorical.*
4. *$(T_{\mathcal{A}}, \prec)$ is relatively computably categorical*

Homogeneous Trees and Equivalence Structures

Theorem

Let $\mathcal{A} = (A, E_1, \ldots, E_n)$ be a nested n-equivalence structure and let $E_0 = A \times A$ and E_{n+1} be equality. Then the following are equivalent.

1. *$\mathcal{A}$ is ultrahomogeneous.*
2. *For each $i \leq n$ there exists k_i such that every E_i class is partitioned into k_i many E_{i+1} classes.*
3. *$T_{\mathcal{A}}$ is ultrahomogeneous in the predecessor representation.*

Corollary

If $\mathcal{A} = (A, E_1, \ldots, E_n)$ is a nested ultrahomogeneous equivalence structure such that all equivalence classes are finite, then $\mathcal{A}$ is ultrahomogeneous if and only if each (A, E_i) is ultrahomogeneous.

Weakly Ultrahomogeneous Structures

Theorem

Let $\mathcal{A} = (A, E_1, \ldots, E_n)$ be a nested n-equivalence structure. Then $\mathcal{A}$ is weakly ultrahomogeneous iff $T_{\mathcal{A}}$ is weakly ultrahomogeneous in the predecessor representation.

There exist computably categorical nested 2-equivalence structures which are not weakly ultrahomogeneous.

Example

Let $\mathcal{A} = (A, E_1, E_2)$ where E_2 has infinitely many classes of size 2 such that infinitely many E_2-classes are split into E_1-classes of size 1 and infinitely many E_2-classes contain a single E_1-class of size 2.

Abelian p-Groups

Groups of the form

$$\mathcal{G} = \oplus_{i<\omega} Z(p^{n_i}) \oplus \oplus_\alpha Z(p^\infty)$$

for some $\alpha \leq \omega$.
The *character* of such a group $\mathcal{G}$ is

$$\chi(\mathcal{A}) = \{(n,k) : card(\{i : n_i = n\}) \geq k\}.$$

$\mathcal{G}$ has *bounded character* if for some finite b and all $(n,k) \in \chi(\mathcal{G})$, $n \leq b$

Computably Categorical p-Groups

Theorem (Goncharov, Smith)

A computable Abelian p-group $\mathcal{G}$ is computably categorical if and only if either

1. $\mathcal{G} \approx \oplus_\alpha Z(p^\infty) \oplus \mathcal{F}$, *where* $\alpha \leq \omega$, *or*
2. $\mathcal{G} \approx \oplus_r Z(p^\infty) \oplus \oplus_\omega Z(p^m) \oplus \mathcal{F}$, *where* $\mathcal{F}$ *is a finite Abelian p-group and* $r, m \in \omega$.

p-groups are locally finite,
so only these could be weakly ultrahomogeneous

Ultrahomogeneous p-Groups

Ultrahomogeneous Abelian p-groups are either divisible, that is, of the form $\oplus_\alpha Z(p^\infty)$ for some $\alpha \leq \omega$,

or *homocyclic*, that is, of the form $\oplus_\omega Z(p^n))$, for some fixed n. (See Cherlin and Felgner 1991)

Thus every countable ultrahomogeneous Abelian p-group has a computable copy

Even the finite group $Z(2) \oplus Z(4)$ is not ultrahomogeneous

The key here is that $(1, 0)$ and $(0, 2)$ both have order two but only $(0, 2)$ is divisible

Some non-WUH p-Groups

THEOREM: For any $m < n$, $\oplus_\omega Z(p^n) \oplus \mathbb{Z}(p^m)$ is not weakly ultrahomogeneous.

PROOF: Let $\{a_1, \ldots, a_k\}$ be a finite exceptional set. Let a be an element of order p^{n-1} in some component not in the support of any a_i and let $b \in Z(p^m)$ have order p^m.

Then a and $a + b$ are both of order p^{n-1} the map taking a to $a + b$ and each a_i to a_i is an isomorphism.

But this can not be extended to an automorphism of $\mathcal{G}$, since a is divisible, whereas $a + b$ is not divisible.

More non-WUH p-groups

Theorem

For any finite $m > 0$, $Z(p^\infty) \oplus Z(p^m)$ is not weakly ultrahomogeneous.

Lemma

For any prime p and any m, n, r with $2m + r < n$, there is an isomorphism ϕ between two subgroups of $Z(p^n) \oplus Z(p^m)$ which fixes every element of order $\leq p^{m+r}$ (hence fixes all elements of $0 \oplus Z(p^m)$ and also $< p^{m+1} > \oplus 0$), but which cannot be extended to an automorphism.

Characterization of WUH

Theorem

A countable Abelian p-group $\mathcal{G}$ is weakly ultrahomogenous if and only if it has one of the following forms:

1. $\mathcal{G} = \oplus_\alpha Z(p^\infty)$ *for some* $\alpha \leq \omega$.
2. $\mathcal{G} = \oplus_{i<\omega} Z(p^n) \oplus \mathcal{F}$, *where n is finite and* $\mathcal{F}$ *is a finite product of cylic groups each having order* $\geq n$.

In the second case, an exceptional set may be given to contain a generator for each factor of $\mathcal{F}$.

Boolean Algebras

Proposition

For the language $(\leq, 0, 1)$ *and a Boolean algebra* $\mathcal{B}$*,*

1. $\mathcal{B}$ *is ultrahomogeneous if and only if* $\mathcal{B}$ *is finite with at most* 4 *elements.*
2. $\mathcal{B}$ *is weakly ultrahomogeneous if and only if* $\mathcal{B}$ *is finite.*

Theorem

For the language $(\wedge, \vee, \neg, 0, 1)$ *and a Boolean algebra* $\mathcal{B}$*,*

1. $\mathcal{B}$ *is ultrahomogeneous iff* $\mathcal{B}$ *is dense*
2. $\mathcal{B}$ *is weakly ultrahomogeneous iff* $\mathcal{B}$ *has finitely many atoms, which is iff* $\mathcal{B}$ *is computably categorical.*

Current and Future Work

Classification of weakly ultrahomogeneous structures, such as

Torsion-free Groups

Trees of transfinite height

THANK YOU

HAPPY BIRTHDAY ROD